A Concrete Introduction to Abstract (Co)Inductive Datatypes

TABLEAUX 2015

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Overview

Part I: Datatypes

Part II: Codatatypes

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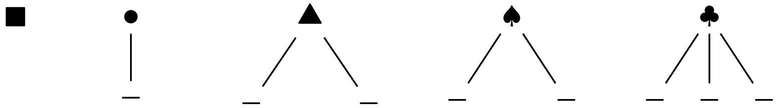
Recall: It's All About Shape and Content

Shapes



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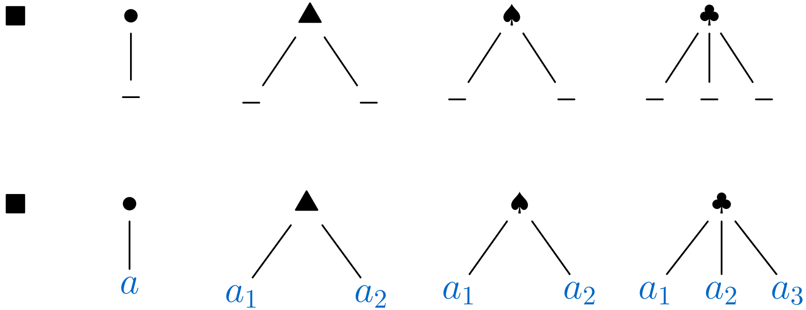

Shapes filled with content from a set $A = \{a_1, a_2, \ldots\}$

Recall: Natural Functors on Set

Set = the class of all sets

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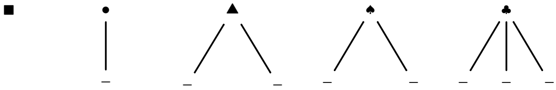

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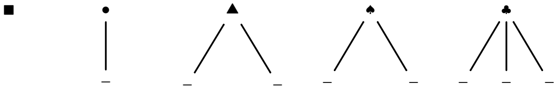

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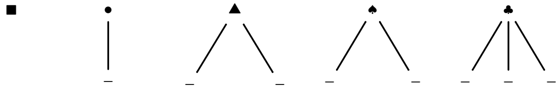

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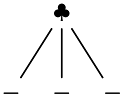

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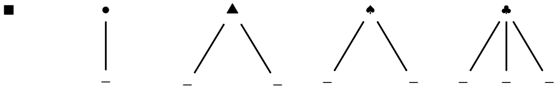

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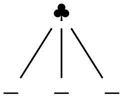

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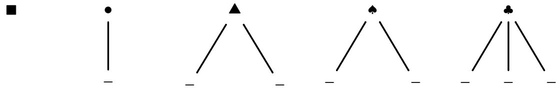

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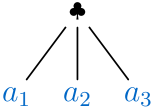

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Recall: Examples of Natural Functors

$\mathsf{F}\ A = \mathbb{N} \times A$

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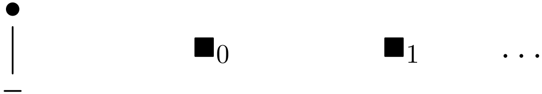
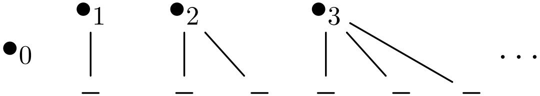


$\mathsf{F}\ A = \mathsf{List}\ A$



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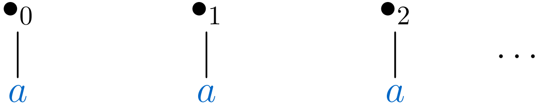

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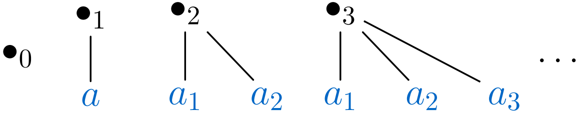

Functorial Action (Mapper)

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$

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A \arrow[d, "f"] & \mathsf{F}\,A \arrow[d, "\mathsf{F}\,f"] \\
B & \mathsf{F}\,B
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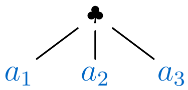
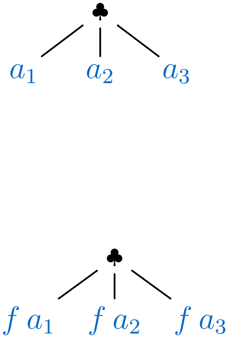


Functorial Action (Mapper)

$$A \xrightarrow{f} B \qquad \mathsf{F}\,A \xrightarrow{\mathsf{F}\,f} \mathsf{F}\,B$$



Keep the same shape

Apply f to the content

Atoms

$$\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$$

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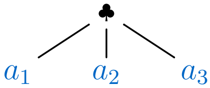

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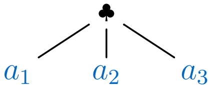

$\{a_1, a_2, a_3\}$

$$\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$$

Natural Functors

$\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

Functoriality: For all $A \xrightarrow{f} B$, we have $\mathsf{F}\,A \xRightarrow{\mathsf{F}\,f} \mathsf{F}\,B$ such that:

$$\mathsf{F}\,\mathsf{id}_A = \mathsf{id}_{\mathsf{F}\,A}$$
$$\mathsf{F}\,(g \circ f) = \mathsf{F}\,g \circ \mathsf{F}\,f$$

Naturality: For all A, we have $\mathsf{F}\,A \xRightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$ such that, for all $A \xrightarrow{f} B$:

$$\mathsf{image}\,f \circ \mathsf{Fatoms}_A = \mathsf{Fatoms}_B \circ \mathsf{image}\,f$$

Examples

$$A \overset{f}{\Rightarrow} B \qquad \mathsf{F}\ A \overset{\mathsf{F}\,f}{\Rightarrow} \mathsf{F}\ B \qquad \mathsf{F}\ A \overset{\mathsf{Fatoms}}{\Rightarrow} \mathcal{P}\ A$$

$\mathsf{F}\ A = \mathbb{N} \times A$

$$\mathsf{F}\,f\ (n, a) = (n, f\ a)$$
$$\mathsf{Fatoms}\ (n, a) = \{a\}$$

$\mathsf{F}\ A = \mathbb{N} + A$

$$\mathsf{F}\,f\ (\mathsf{Left}\ n) = \mathsf{Left}\ n \qquad \mathsf{F}\,f\ (\mathsf{Right}\ a) = \mathsf{Right}\ (f\ a)$$
$$\mathsf{Fatoms}\ (\mathsf{Left}\ n) = \varnothing \qquad \mathsf{Fatoms}\ (\mathsf{Right}\ a) = \{a\}$$

$\mathsf{F}\ A = \mathsf{List}\ A$

$$\mathsf{F}\,f\ (a_1 \cdot a_2 \cdot \ldots \cdot a_n) = f\ a_1 \cdot f\ a_2 \ldots \cdot f\ a_n$$
$$\mathsf{Fatoms}\ (a_1 \cdot a_2 \cdot \ldots \cdot a_n) = \{a_1, a_2, \ldots, a_n\}$$

Recall: Iterating Shape Composition

Natural functor $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

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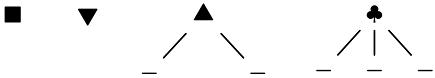
Natural functor $F : \mathsf{Set} \to \mathsf{Set}$

The shapes of F: 

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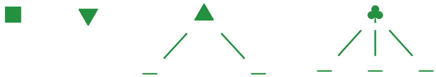

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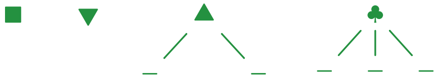

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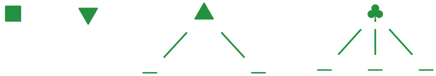
Copies of the shapes of F: 

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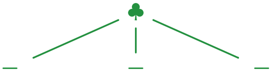

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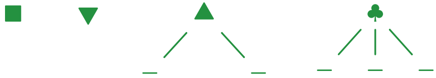
Copies of the shapes of F: 

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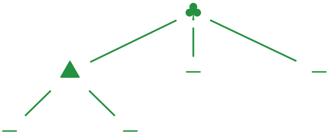

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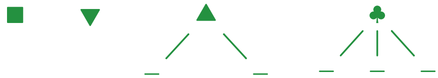

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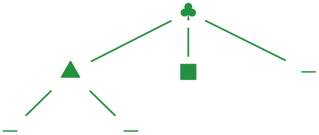

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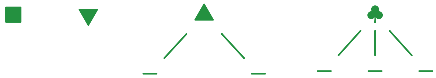

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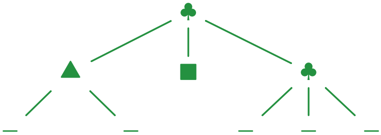

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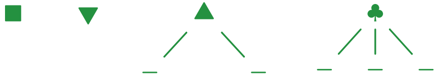

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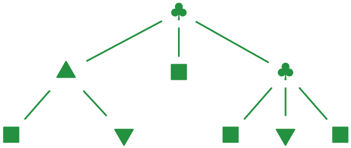

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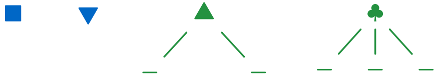

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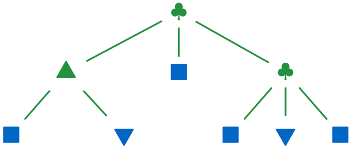

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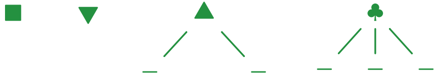

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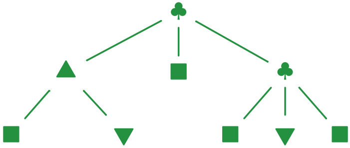

Define I_F = the set of all such finitary couplings

Recall: Properties of $\mathsf{I_F}$

Given a natural functor F, $(\mathsf{I_F},\ \mathsf{ctor} : \mathsf{F}\ \mathsf{I_F} \to \mathsf{I_F})$ satisfies:

ctor bijection

$$\boxed{\mathsf{I_F} = \text{the datatype of F}}$$

Iteration (Initial Algebra Property): For all $(A, s : \mathsf{F}\ A \to A)$, there exists a unique function iter_s such that

$$\begin{tikzcd} \mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{F}\ \mathsf{iter}_s"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"] \\ \mathsf{I_F} \arrow[r, "\mathsf{iter}_s"'] & A \end{tikzcd}$$

Induction: Given any predicate φ on $\mathsf{I_F}$

$$\frac{\forall x \in \mathsf{F}\ \mathsf{I_F}.\ (\forall \mathsf{i} \in \mathsf{Fatoms}\ \mathsf{x}.\ \varphi\ \mathsf{i}) \Rightarrow \varphi\ (\mathsf{ctor}\ \mathsf{x})}{\forall i \in \mathsf{I_F}.\ \varphi\ \mathsf{i}}$$

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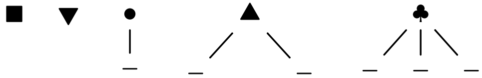

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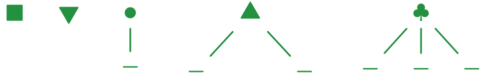

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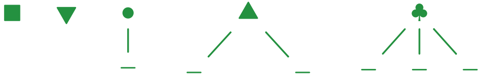
Copies of the shapes of F: 

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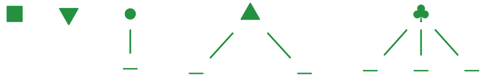
Copies of the shapes of F: 

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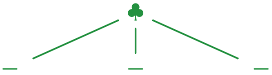

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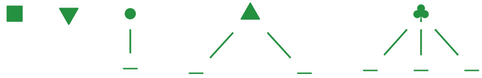
Copies of the shapes of F: 

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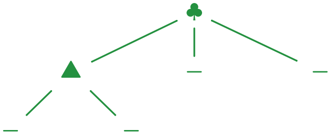

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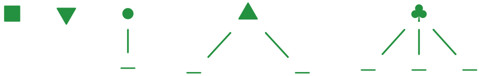
Copies of the shapes of F: 

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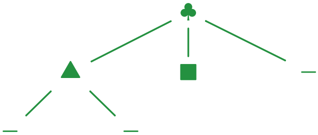

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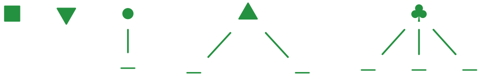

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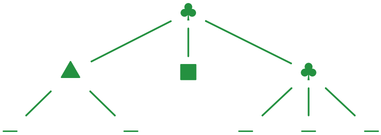

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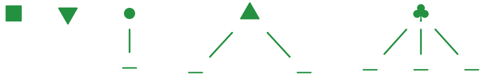

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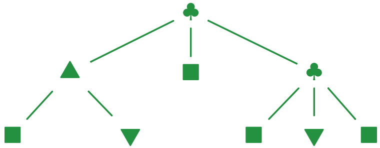

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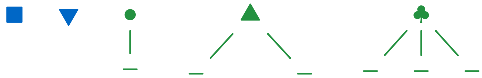

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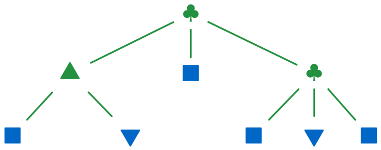

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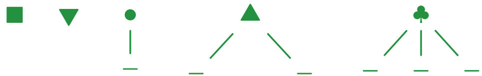
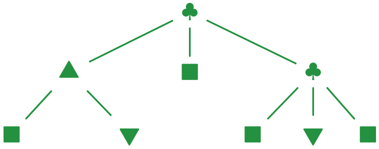


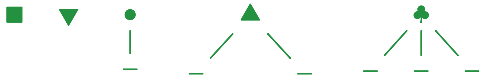
Put them together by plugging in shape for content slot until there are no lingering slots left!



The leaves are always empty-content shapes

Coiterating Shape Composition

Natural functor $F : \mathsf{Set} \to \mathsf{Set}$

Copies of the shapes of F: 

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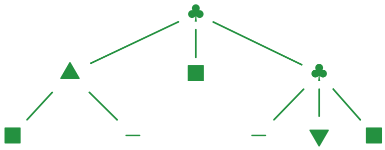
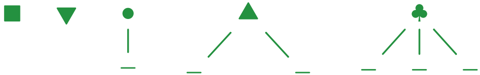


Allow infinite couplings

Coiterating Shape Composition

Natural functor $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

Copies of the shapes of F: 

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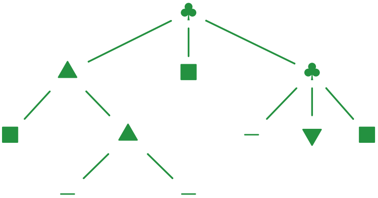

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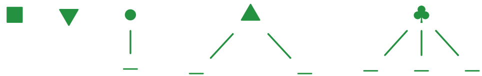
Copies of the shapes of F: 

Put them together by plugging in shape for content slot
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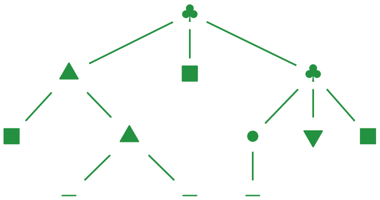

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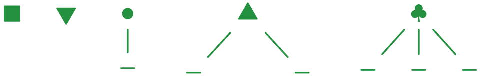

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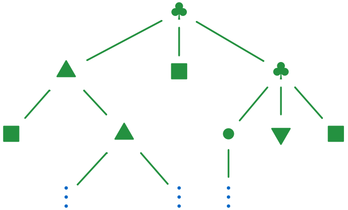
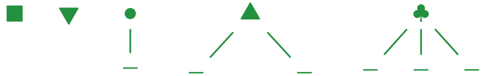


Allow infinite couplings

Coiterating Shape Composition

Natural functor $F : \mathsf{Set} \to \mathsf{Set}$

Copies of the shapes of F: 

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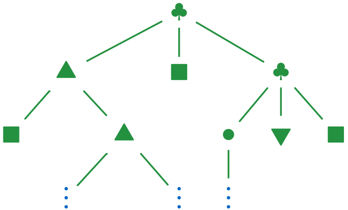
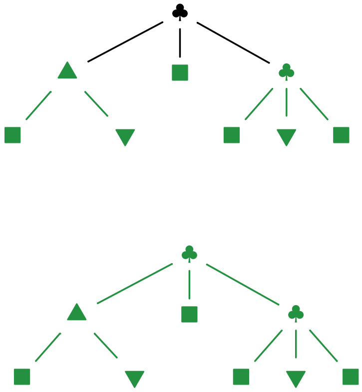


Define J_F = the set of all such (possibly) infinitary couplings

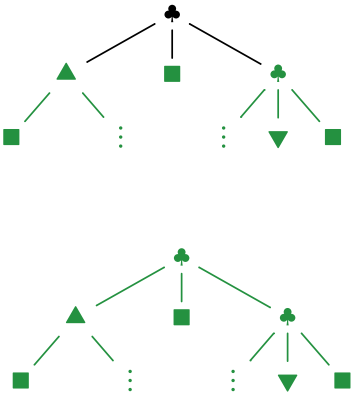
Recall: Properties of I_F: Bijectivity



$$\begin{tikzcd} F\, I_F \arrow[d, shift left, "\mathrm{ctor}"] \\ I_F \arrow[u, shift left, "\mathrm{dtor}"] \end{tikzcd}$$

ctor and dtor are mutually inverse bijections

Properties of J_F: Bijectivity



$$\begin{tikzcd} F\,J_F \arrow[d, shift left, "\mathrm{ctor}"] \\ J_F \arrow[u, shift left, "\mathrm{dtor}"] \end{tikzcd}$$

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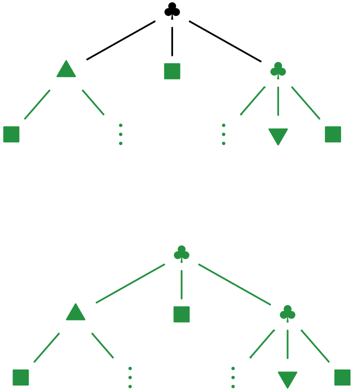
Properties of J_F: Bijectivity



$$\begin{tikzcd} F\, J_F \arrow[d, shift left, "\mathrm{ctor}"] \\ J_F \arrow[u, shift left, "\mathrm{dtor}"] \end{tikzcd}$$

ctor and dtor are mutually inverse bijections

A similar property holds for J_F, where we use the same notations for constructor and destructor

I_F is embedded in J_F

$$
\begin{tikzcd}
\mathsf{F}\,\mathsf{I}_\mathsf{F} \arrow[r, "\mathsf{F}\,\iota"] \arrow[d, shift left, "\mathsf{ctor}"] & \mathsf{F}\,\mathsf{J}_\mathsf{F} \arrow[d, shift left, "\mathsf{ctor}"] \\
\mathsf{I}_\mathsf{F} \arrow[r, "\iota"] \arrow[u, shift left, "\mathsf{dtor}"] & \mathsf{J}_\mathsf{F} \arrow[u, shift left, "\mathsf{dtor}"]
\end{tikzcd}
$$

$\mathsf{I_F}$ is embedded in $\mathsf{J_F}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{F}\,\mathsf{I_F} \arrow[r, "\mathsf{F}\,\iota"] \arrow[d, shift left, "\mathsf{ctor}"] & \mathsf{F}\,\mathsf{J_F} \arrow[d, shift left, "\mathsf{ctor}"] \\
\mathsf{I_F} \arrow[r, "\iota"] \arrow[u, shift left, "\mathsf{dtor}"] & \mathsf{J_F} \arrow[u, shift left, "\mathsf{dtor}"]
\end{tikzcd}$$

$\iota = \mathsf{iter}_{\mathsf{ctor}:\mathsf{F}\ \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F}}$

Properties of $\mathsf{J_F}$: Coiteration

$$
\begin{tikzcd}
\mathsf{F}\, A & \\
A \arrow[u, "s"'] & \mathsf{J_F}
\end{tikzcd}
$$

Properties of J_F: Coiteration

$$
\begin{tikzcd}
\mathsf{F}\,A & \\
A \arrow[u, "s"] \arrow[r, dotted, "f"] & \mathsf{J}_\mathsf{F}
\end{tikzcd}
$$

Properties of $\mathsf{J_F}$: Coiteration

$$\begin{tikzcd}
\mathsf{F}\,A & \\
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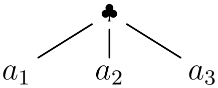
\end{tikzcd}$$

a

Properties of $\mathsf{J_F}$: Coiteration



$$
\begin{tikzcd}
\mathsf{F}\,A & \\
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\end{tikzcd}
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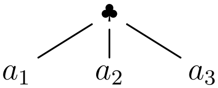
a

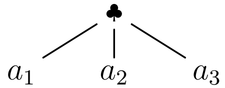
Properties of $\mathsf{J_F}$: Coiteration



$$\begin{tikzcd}
\mathsf{F}\,A & \mathsf{F}\,\mathsf{J_F} \arrow[d, "\mathsf{ctor}"'] \\
A \arrow[u, "s"'] \arrow[r, dotted, "f"] & \mathsf{J_F}
\end{tikzcd}$$

a

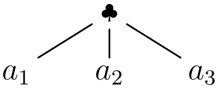
Properties of J_F: Coiteration

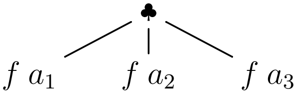


$$\begin{tikzcd}
\mathsf{F}\ A \arrow[r, dotted, "\mathsf{F}\ f"'] & \mathsf{F}\ \mathsf{J}_\mathsf{F} \arrow[d, "\mathsf{ctor}"'] \\
A \arrow[u, "s"'] \arrow[r, dotted, "f"] & \mathsf{J}_\mathsf{F}
\end{tikzcd}$$

a

Properties of $\mathsf{J_F}$: Coiteration





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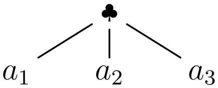

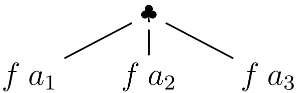

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\end{tikzcd}$$

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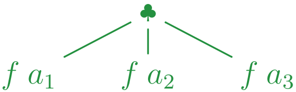
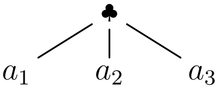
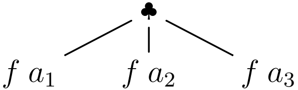


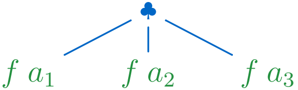
Properties of $\mathsf{J_F}$: Coiteration





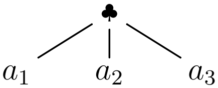
$$
\begin{tikzcd}
\mathsf{F}\ A \arrow[r, dotted, "\mathsf{F}\ f"'] & \mathsf{F}\ \mathsf{J_F} \arrow[d, "\mathsf{ctor}"'] \\
A \arrow[u, "s"'] \arrow[r, dotted, "f"] & \mathsf{J_F}
\end{tikzcd}
$$

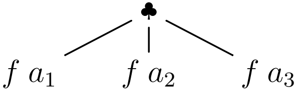
a



a_1, a_2, a_3 are not “smaller” than a in any sense

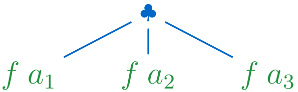
Properties of $\mathsf{J_F}$: Coiteration





$$\begin{tikzcd}
\mathsf{F}\ A \arrow[r, dotted, "\mathsf{F}\ f"'] & \mathsf{F}\ \mathsf{J_F} \arrow[d, "\mathsf{ctor}"'] \\
A \arrow[u, "s"'] \arrow[r, dotted, "f"] & \mathsf{J_F}
\end{tikzcd}$$

a



a_1, a_2, a_3 are not "smaller" than a in any sense
But computation has made progress

Properties of $\mathsf{J_F}$: Coiteration

$$\begin{tikzcd} & \mathsf{F}\ A \\ A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J_F} \end{tikzcd}$$

Properties of $\mathsf{J_F}$: Coiteration

$$
\begin{tikzcd}
& \mathsf{F}\ A \\
A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J_F}
\end{tikzcd}
$$

a

Properties of J_F: Coiteration

$s\ a$

$$\begin{tikzcd}
 & \mathsf{F}\ A \\
A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J}_\mathsf{F}
\end{tikzcd}$$

a

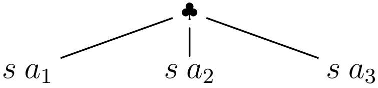
Properties of $\mathsf{J_F}$: Coiteration

$$
\begin{tikzcd}[column sep=small]
 & \clubsuit \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
a_1 & a_2 & a_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathsf{F}\, A \\
A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J_F}
\end{tikzcd}
$$

a

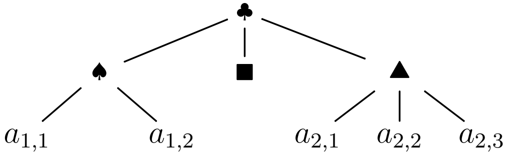
Properties of J_F: Coiteration



$$\begin{tikzcd}
 & \mathsf{F}\ A \\
A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J}_\mathsf{F}
\end{tikzcd}$$

a

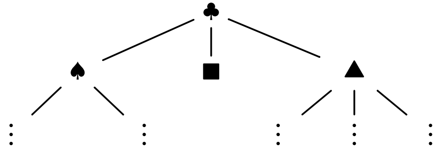
Properties of $\mathsf{J_F}$: Coiteration



$$\begin{tikzcd}
 & \mathsf{F}\ A \\
A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J_F}
\end{tikzcd}$$

a

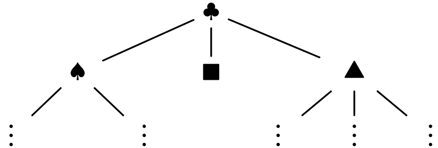
Properties of J_F: Coiteration



$$\begin{tikzcd} & \mathsf{F}\,A \\ A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J}_\mathsf{F} \end{tikzcd}$$

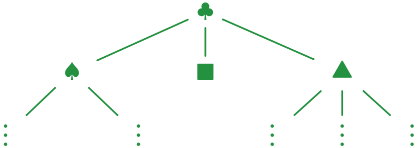
a

Properties of $\mathsf{J_F}$: Coiteration

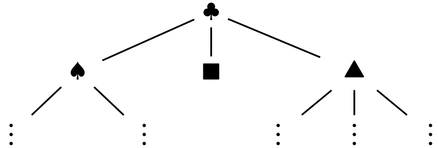


$$\begin{tikzcd} & \mathsf{F}\,A \\ A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J_F} \end{tikzcd}$$

a



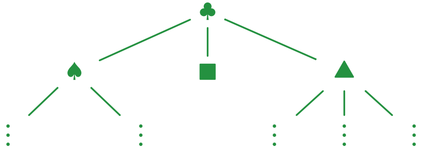
Properties of J_F: Coiteration



$$
\begin{tikzcd}
 & \mathsf{F}\,A \\
A \arrow[ur, "s"] \arrow[r, dotted, "f"] & \mathsf{J}_\mathsf{F}
\end{tikzcd}
$$

$s\ a$ = the seed encoding the growth of the tree $f\ a$

a



Properties of $\mathsf{J_F}$: Coiteration

Given a natural functor F, $\quad (\mathsf{J_F},\ \mathsf{dtor} : \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F})$

Coiteration (Final Coalgebra Property): For all $(A, s : A \to \mathsf{F}\ A)$, there exists a unique function coiter_s with

$$\begin{tikzcd}
\mathsf{F}\ A \arrow[r, "\mathsf{F}\ \mathsf{coiter}_s"] & \mathsf{F}\ \mathsf{J_F} \arrow[d, "\mathsf{ctor}"] \\
A \arrow[u, "s"] \arrow[r, "\mathsf{coiter}_s"] & \mathsf{J_F}
\end{tikzcd}$$

Properties of $\mathsf{J_F}$: Coiteration

Given a natural functor F, $\quad (\mathsf{J_F},\ \mathsf{dtor} : \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F})$

Coiteration (Final Coalgebra Property): For all $(A, s : A \to \mathsf{F}\ A)$, there exists a unique function coiter_s with

$$\begin{tikzcd}
\mathsf{F}\ A \arrow[r, "\mathsf{F}\ \mathsf{coiter}_s"] & \mathsf{F}\ \mathsf{J_F} \\
A \arrow[u, "s"] \arrow[r, "\mathsf{coiter}_s"] & \mathsf{J_F} \arrow[u, "\mathsf{dtor}"']
\end{tikzcd}$$

Properties of $\mathsf{J_F}$: Coiteration

Given a natural functor F, $(\mathsf{J_F},\ \mathsf{dtor} : \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F})$

Coiteration (Final Coalgebra Property): For all $(A, s : A \to \mathsf{F}\ A)$, there exists a unique function coiter_s with

$$
\begin{tikzcd}
\mathsf{F}\ A \arrow[r, "\mathsf{F}\ \mathsf{coiter}_s"] & \mathsf{F}\ \mathsf{J_F} \\
A \arrow[u, "s"] \arrow[r, "\mathsf{coiter}_s"] & \mathsf{J_F} \arrow[u, "\mathsf{dtor}"']
\end{tikzcd}
$$

$\boxed{\mathsf{J_F} = \text{the codatatype of } \mathsf{F}}$

The $\mathsf{I_F}$ to $\mathsf{J_F}$ embedding revisited

$$\begin{tikzcd}
\mathsf{F\ I_F} \arrow[rr, "\mathsf{F}\ \iota"] \arrow[d, shift left, "\mathsf{ctor}"] & & \mathsf{F\ J_F} \arrow[d, shift left, "\mathsf{ctor}"] \\
\mathsf{I_F} \arrow[u, shift left, "\mathsf{dtor}"] \arrow[rr, "\iota"] & & \mathsf{J_F} \arrow[u, shift left, "\mathsf{dtor}"]
\end{tikzcd}$$

ι can be regarded as defined by iteration on $\mathsf{I_F}$

$\iota = \mathsf{iter}_{\mathsf{ctor}}$

The $\mathsf{I_F}$ to $\mathsf{J_F}$ embedding revisited

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathsf{F}\,\mathsf{I_F} \arrow[r, "\mathsf{F}\,\iota"] \arrow[d, shift left, "\mathsf{ctor}"] & \mathsf{F}\,\mathsf{J_F} \arrow[d, shift left, "\mathsf{ctor}"] \\
\mathsf{I_F} \arrow[r, "\iota"] \arrow[u, shift left, "\mathsf{dtor}"] & \mathsf{J_F} \arrow[u, shift left, "\mathsf{dtor}"]
\end{tikzcd}$$

ι can be regarded as defined by
iteration on $\mathsf{I_F}$ but also by coiteration on $\mathsf{J_F}$!

$\iota = \mathsf{iter}_{\mathsf{ctor}} = \mathsf{coiter}_{\mathsf{dtor}}$

Properties of $\mathsf{J_F}$: Coinduction

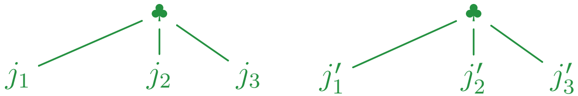
j j'

Properties of $\mathsf{J_F}$: Coinduction

j j'

Want: $j = j'$

Properties of $\mathsf{J_F}$: Coinduction



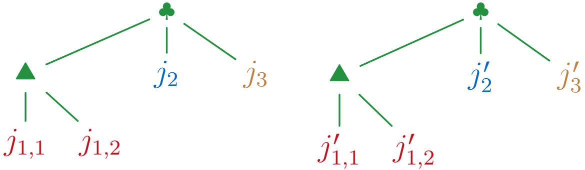
Want: $j = j'$

Properties of $\mathsf{J_F}$: Coinduction

Suffices:

$$j_1 = j_1'$$
$$j_2 = j_2'$$
$$j_3 = j_3'$$

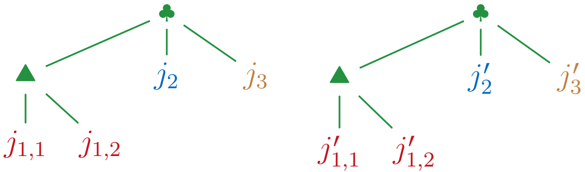
Properties of J_{F}: Coinduction



Suffices:

$$j_1 = j'_1$$
$$j_2 = j'_2$$
$$j_3 = j'_3$$

Properties of $\mathsf{J_F}$: Coinduction

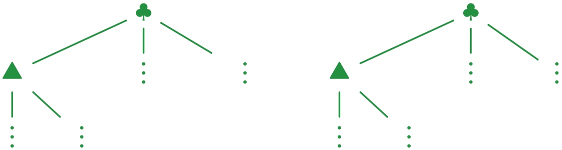


Suffices: $j_{1,1} = j'_{1,1},\ j_{1,2} = j'_{1,2}$

$$j_2 = j'_2$$

$$j_3 = j'_3$$

Properties of $\mathsf{J_F}$: Coinduction



Suffices: $j_{1,1} = j'_{1,1},\ j_{1,2} = j'_{1,2}$
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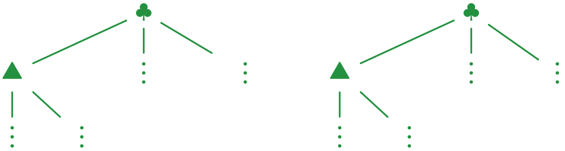
Properties of $\mathsf{J_F}$: Coinduction



If we can stay in the game indefinitely, then equality holds!

Suffices: $$\begin{gathered} j_{1,1} = j'_{1,1},\ j_{1,2} = j'_{1,2} \\ j_2 = j'_2 \\ j_3 = j'_3 \end{gathered}$$

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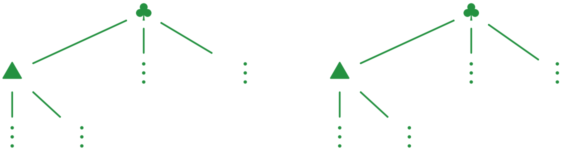

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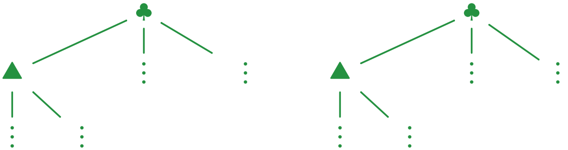
Properties of $\mathsf{J_F}$: Coinduction



If we can stay in the game indefinitely, then equality holds!
But how to show we can "stay in the game"?
By exhibiting a "strategy"

Suffices: $j_{1,1} = j'_{1,1},\ j_{1,2} = j'_{1,2}$
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But First: Relators

$$\begin{tikzcd} A \arrow[d, dash, "R"] \\ B \end{tikzcd}$$

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\end{tikzcd}
$$

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B
\end{tikzcd}
\qquad\qquad
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Two elements of $\mathsf{F}\,A$ and $\mathsf{F}\,B$ are related by $\mathsf{Frel}\,R$ iff

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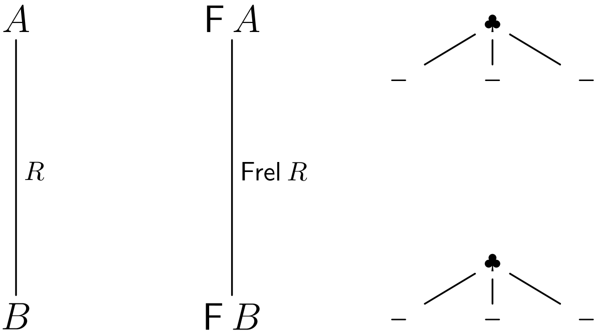

Two elements of $\mathsf{F}\ A$ and $\mathsf{F}\ B$ are related by $\mathsf{Frel}\ R$ iff they have the same shape

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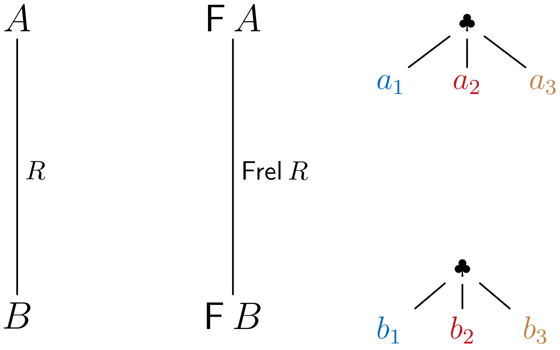

Two elements of $\mathsf{F}\,A$ and $\mathsf{F}\,B$ are related by $\mathsf{Frel}\,R$ iff
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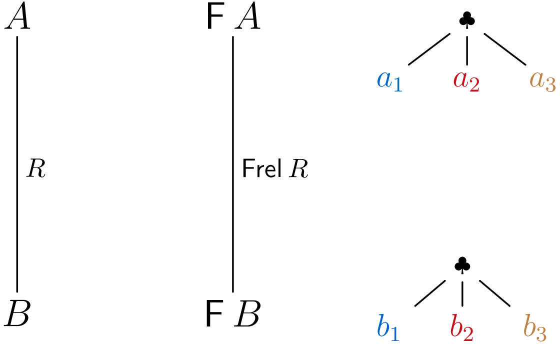
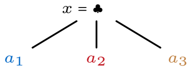


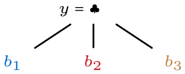
Two elements of F A and F B are related by Frel R iff
they have the same shape
and the contents from corresponding slots are related by R
$R\, a_1\, b_1$, $R\, a_2\, b_2$, $R\, a_3\, b_3$

Relator Defined from Mapper

R relation between A and B, $x \in \mathsf{F}\ A$, $y \in \mathsf{F}\ B$

Relator Defined from Mapper





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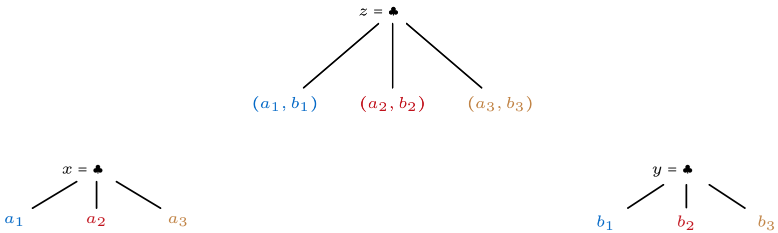

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Relators for the Running Examples

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R relation between A and B
Frel R relation between F A and F B

F $A = \mathbb{N} \times A$ Frel $R\ (m, a)\ (n, b) \Leftrightarrow (m = n \wedge R\ a\ b)$

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$(\exists n.\ u = v = \text{Left}\ n) \vee$
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F A = List A

Relators for the Running Examples

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$\mathsf{F}\ A = \mathsf{List}\ A$ $\quad$ $\mathsf{Frel}\ R\ (a_1 \cdot a_2 \cdot \ldots \cdot a_m)\ \ (b_1 \cdot b_2 \cdot \ldots \cdot b_n) \Leftrightarrow$

Relators for the Running Examples

R relation between A and B
Frel R relation between F A and F B

$\mathsf{F}\,A = \mathbb{N} \times A$	$\mathsf{Frel}\,R\,(m, a)\,(n, b) \Leftrightarrow (m = n \,\wedge\, R\,a\,b)$
$\mathsf{F}\,A = \mathbb{N} + A$	$\mathsf{Frel}\,R\,u\,v \Leftrightarrow$ $(\exists n.\ u = v = \mathsf{Left}\,n) \vee$ $(\exists a, b.\ u = \mathsf{Right}\,a \,\wedge\, v = \mathsf{Right}\,b \,\wedge\, R\,a\,b)$
$\mathsf{F}\,A = \mathsf{List}\,A$	$\mathsf{Frel}\,R\,(a_1 \cdot a_2 \cdot \ldots \cdot a_m)\ (b_1 \cdot b_2 \cdot \ldots \cdot b_n) \Leftrightarrow$ $m = n \,\wedge\, (\forall i.\ R\,a_i\,b_i)$

Back to the “Strategy” for Proving Equality

$$\begin{tikzcd} \mathsf{J_F} \arrow[d, "\mathsf{dtor}"] \\ \mathsf{F\ J_F} \end{tikzcd}$$

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$$\begin{tikzcd} \mathsf{J_F} \arrow[d, "\mathsf{dtor}"] \\ \mathsf{F}\ \mathsf{J_F} \end{tikzcd}$$

Given binary relation R on $\mathsf{J_F}$

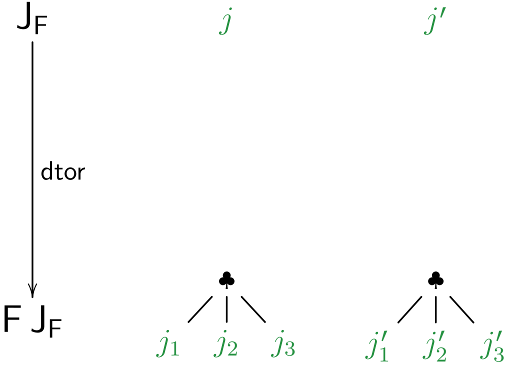
Back to the "Strategy" for Proving Equality

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{J_F} \arrow[d, "\mathsf{dtor}"] & {\color{green} j} & {\color{green} j'} \\
\mathsf{F}\ \mathsf{J_F} & &
\end{tikzcd}$$

Given binary relation R on $\mathsf{J_F}$

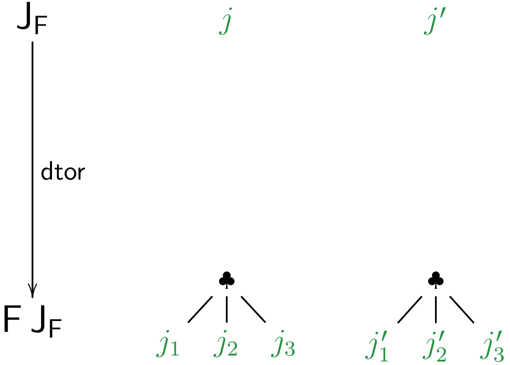
If $\forall j, j'.\, R\; j\; j'$

Back to the “Strategy” for Proving Equality



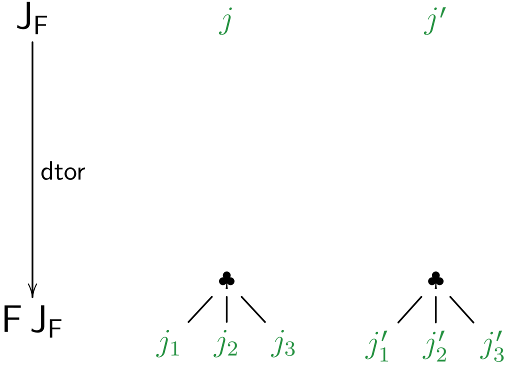
Given binary relation R on $\mathsf{J_F}$
If $\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')$

Back to the “Strategy” for Proving Equality



Given binary relation R on $\mathsf{J_F}$
If $\forall j, j'.\, R\; j\; j' \Rightarrow \mathsf{Frel}\; R\; (\mathsf{dtor}\; j)\; (\mathsf{dtor}\; j')$
Then R is included in equality

Back to the “Strategy” for Proving Equality



Given binary relation R on J_F

If $\forall j, j'.\ R\ j\ j' \Rightarrow \text{Frel}\ R\ (\text{dtor}\ j)\ (\text{dtor}\ j')$

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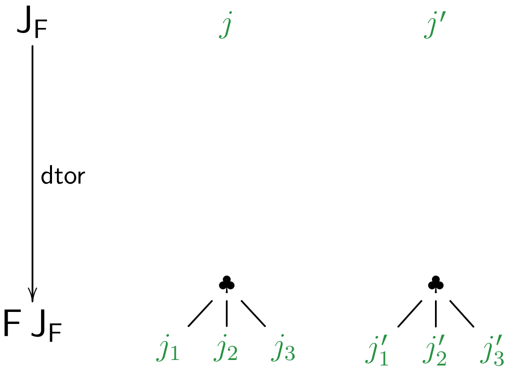
Back to the “Strategy” for Proving Equality



Given binary relation R on J_F

If $\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')$ $\boxed{R \text{ F-bisimulation}}$

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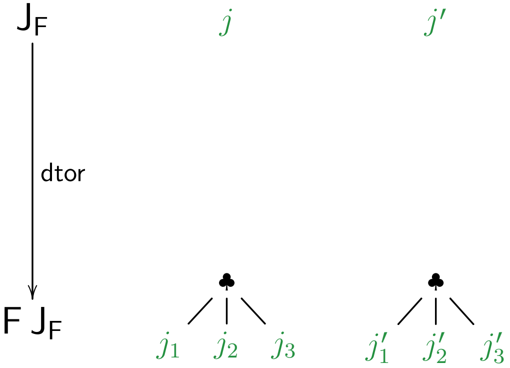

Summary: to prove $j = j'$,

Given binary relation R on $\mathsf{J_F}$

If $\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')$ $\boxed{R \text{ F-bisimulation}}$

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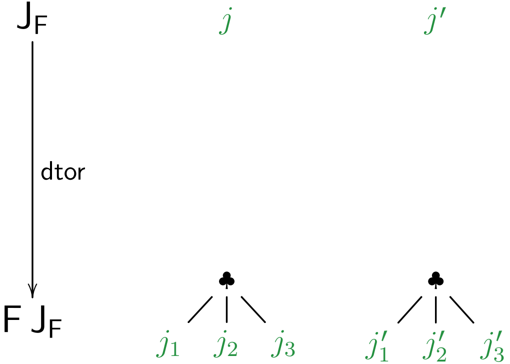
Back to the “Strategy” for Proving Equality



Summary: to prove $j = j'$, find F-bisimulation R with $R\ j\ j'$

Given binary relation R on J_F

If $\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')$ $\boxed{R \text{ F-bisimulation}}$

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Summary for J_F

Given a natural functor F, $\quad (J_F,\ \mathrm{dtor} : J_F \to F\ J_F)$ satisfies:

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Summary for $\mathsf{J_F}$

Given a natural functor F, $(\mathsf{J_F},\ \mathsf{dtor} : \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F})$ satisfies:

dtor bijection

Coiteration (Final Coalgebra Property): For all $(A, s : A \to \mathsf{F}\ A)$, there exists a unique function coiter_s with

$$\begin{tikzcd}
\mathsf{F}\ A \arrow[r, "\mathsf{F}\ \mathsf{coiter}_s"] & \mathsf{F}\ \mathsf{J_F} \\
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Coinduction: Given any binary relation R on $\mathsf{J_F}$

$$\frac{R \text{ is an F-bisimulation}}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

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Coinduction: Given any binary relation R on $\mathsf{J_F}$

$$\frac{\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

Summary for $\mathsf{J_F}$

Given a natural functor F, $(\mathsf{J_F},\ \mathsf{dtor} : \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F})$ satisfies:

dtor bijection $\boxed{\mathsf{J_F} = \text{the codatatype of F}}$

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Example of Codatatype

Let B be a fixed set. $\mathsf{F}\ A = B \times A$

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The shapes of F: $(b, \_)$ for each $b \in B$

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Who is J_{F}?

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Its elements have the form $(b_1, (b_2, \ldots, (b_n, \ldots$
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Who is J_F?
Its elements have the form $(b_1, (b_2, \ldots, (b_n, \ldots$
I.e., essentially streams $b_1 \cdot b_2 \cdot \ldots \cdot b_n \cdot \ldots$
So $\mathsf{J}_\mathsf{F} = \mathsf{Stream}_\mathsf{B}$

Example of Codatatype: Stream

B fixed $\quad \mathsf{F}\ A = B \times A \quad f = \mathsf{coiter}_s \quad \mathsf{J_F} = \mathsf{Stream_B}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & \mathsf{J_F} \arrow[d, "\mathsf{dtor}"] \\ \mathsf{F}\ A \arrow[r, "\mathsf{F}\ f"] & \mathsf{F}\ \mathsf{J_F} \end{tikzcd}$$

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Example of Codatatype: Stream

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B \times A \arrow[r, "B\times f"] & B \times \mathsf{J_F}
\end{tikzcd}$$

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Define: $\mathsf{hd} = \pi_1 \circ \mathsf{dtor} \quad \mathsf{tl} = \pi_2 \circ \mathsf{dtor}$
$\mathsf{hd}^A = \pi_1 \circ s \quad \mathsf{tl}^A = \pi_2 \circ s$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & \mathsf{J_F} \arrow[d, "\mathsf{dtor}"] \\ B \times A \arrow[r, "B \times f"] & B \times \mathsf{J_F} \end{tikzcd}$$

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$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\langle \mathsf{hd}^A, \mathsf{tl}^A \rangle"'] & \mathsf{J_F} \arrow[d, "\langle \mathsf{hd}, \mathsf{tl} \rangle"] \\
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$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[drr, "\mathsf{hd}^A"'] & & \mathsf{J_F} \arrow[d, "\mathsf{hd}"] \\
& & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathsf{tl}^A"'] & \mathsf{J_F} \arrow[d, "\mathsf{hd}"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

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$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\mathsf{hd}^A"'] & \mathsf{J_F} \arrow[d, "\mathsf{hd}"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathsf{tl}^A"'] & \mathsf{J_F} \arrow[d, "\mathsf{hd}"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$\mathsf{hd}\ (f\ a) = \mathsf{hd}^A\ a$

$\mathsf{tl}\ (f\ a) = f\ (\mathsf{tl}^A\ a)$

Example of Codatatype: Stream

B fixed $\mathsf{F}\,A = B \times A$ $f = \mathsf{coiter}_s$ $\mathsf{J_F} = \mathsf{Stream_B}$

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Standard stream coiteration

Example of Codatatype: Stream

B fixed $\quad \mathsf{F}\ A = B \times A \quad\quad \mathsf{J_F} = \mathsf{Stream_B}$

$$\begin{tikzcd}
\mathsf{J_F} \arrow[rrr, dash, "R"] \arrow[d, "\mathsf{dtor}"'] & & & \mathsf{J_F} \arrow[d, "\mathsf{dtor}"] \\
\mathsf{F}\ \mathsf{J_F} \arrow[rrr, dash, "\mathsf{Frel}\ R"] & & & \mathsf{F}\ \mathsf{J_F}
\end{tikzcd}$$

$$\frac{R \text{ is an F-bisimulation}}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

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B \times \mathsf{J_F} \arrow[rr, dash, "((b,j),(b',j')) \mapsto b=b' \wedge R\ j\ j'"] & & B \times \mathsf{J_F}
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$$

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Concrete Example of Coiteration

$$\begin{aligned}&\mathsf{ev} : \mathsf{Stream}_B \to \mathsf{Stream}_B\\&\quad \mathsf{hd}\,(\mathsf{ev}\ j) = \mathsf{hd}\ j\\&\quad \mathsf{tl}\,(\mathsf{ev}\ j) = \mathsf{ev}\,(\mathsf{tl}\,(\mathsf{tl}\ j))\end{aligned}$$

Concrete Example of Coiteration

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$\quad \mathsf{tl}\,(\mathsf{odd}\ j) = \mathsf{odd}\,(\mathsf{tl}\,(\mathsf{tl}\ j))$

$\mathsf{zip} : \mathsf{Stream}_B \times \mathsf{Stream}_B \to \mathsf{Stream}_B$

$\quad \mathsf{hd}\,(\mathsf{zip}\,(j_1, j_2)) = \mathsf{hd}\ j_1$

$\quad \mathsf{tl}\,(\mathsf{zip}\,(j_1, j_2)) = \mathsf{zip}\,(j_2,\ \mathsf{tl}\ j_1)$

Pattern-Based Incremental Coinduction

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$\mathsf{zip}\,(\mathsf{ev}\,(\mathsf{tl}\,(\mathsf{tl}\ j)), \mathsf{odd}\,(\mathsf{tl}\,(\mathsf{tl}\ j))) = \mathsf{tl}\,(\mathsf{tl}\ j)$ $\quad$ $\mathsf{hd}\ \ldots = \mathsf{hd}\,(\mathsf{tl}\ j)$

Bisimulation: $R\ j_1\ j_2 \equiv$
$j_1 = \mathsf{zip}\,(\mathsf{ev}\ j_2, \mathsf{odd}\ j_2) \lor$
$\exists j.\ j_1 = \mathsf{zip}\,(\mathsf{odd}\ j,\ \mathsf{ev}\,(\mathsf{tl}\,(\mathsf{tl}\ j))) \land j_2 = \mathsf{tl}\ j$

Universe of (Co)Datatypes

Natural functors are a class of functors

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Nesting datatypes in codatatypes or vice versa
allows for modular specs of fancy data structures

Universe of (Co)Datatypes in Isabelle/HOL

The Isabelle system maintains a database of natural functors

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codatatype Stream A = Cons (hd : A) (tl : List A)

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In the background:

- Isabelle parses this into a natural functor: $B \mapsto B \times A$
- Then infers high-level principles for (co)recursion and (co)induction for Stream
- Finally, Stream is itself registered as a natural functor

Examples

datatype List A = Nil | Cons A (List A)

Examples

datatype List A = Nil $\mid$ Cons A (List A)

codatatype Lazy_List A = Nil $\mid$ Cons A (List A)

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datatype List A = Nil $|$ Cons A (List A)

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datatype X A = Leaf A $|$ Node (X A) (X A)

Examples

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datatype BTree A = Leaf A $|$ Node $(X\ A)$ $(X\ A)$

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node $(X\ A)$ $(X\ A)$

datatype $X\ A$ = Node A (List $(X\ A)$)

Examples

datatype List A = Nil $|$ Cons A (List A)

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datatype BTree A = Leaf A $|$ Node $(X\ A)$ $(X\ A)$

datatype Tree A = Node A (List (Tree A))

Examples

datatype List A = Nil | Cons A (List A)

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datatype BTree A = Leaf A | Node (X A) (X A)

datatype Tree A = Node A (List (Tree A))
finite-depths, finitely branching
A-labeled trees

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A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node $(X\ A)$ $(X\ A)$

codatatype Tree A = Node A (Lazy_List (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil $|$ Cons A (List A)

codatatype Lazy_List A = Nil $|$ Cons A (List A)

datatype BTree A = Leaf A $|$ Node (X A) (X A)

codatatype Tree A = Node A (Countable_Set (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil $|$ Cons A (List A)

codatatype Lazy_List A = Nil $|$ Cons A (List A)

datatype BTree A = Leaf A $|$ Node $(X\ A)$ $(X\ A)$

codatatype Tree A = Node A (Set$_k$ (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (Multi_Set (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil $\mid$ Cons A (List A)

codatatype Lazy_List A = Nil $\mid$ Cons A (List A)

datatype BTree A = Leaf A $\mid$ Node $(X\ A)$ $(X\ A)$

codatatype Tree A = Node A (Fuzzy_Set (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil $\mid$ Cons A (List A)

codatatype Lazy_List A = Nil $\mid$ Cons A (List A)

datatype BTree A = Leaf A $\mid$ Node (X A) (X A)

codatatype Tree A = Node A (PLUG_YOUR_OWN (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil $|$ Cons A (List A)

codatatype Lazy_List A = Nil $|$ Cons A (List A)

datatype BTree A = Leaf A $|$ Node $(X\ A)$ $(X\ A)$

codatatype Tree A = Node A (PLUG_YOUR_OWN (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

- Show a set operator to be a (bounded) natural functor
- Register it
- Then Isabelle will allow nesting it in (co)datatype expressions

Examples

datatype X A =
    Elements (Finite_Set (X A))

Examples

datatype Hereditarily_Finite_Set A =
    Elements (Finite_Set (Hereditarily_Finite_Set A))

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datatype Hereditarily_Finite_Set A =
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... in the presence of the Foundation Axiom

Examples

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. . . in the presence of Aczel's Anti-Foundation Axiom

Summary

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Moreover, the abstract constructions have very concrete intuitions

The abstract reality can be very concrete

Relevant Literature

Our work:
Natural functors:
LICS'12, ESOP'15
Flexible corecursion:
ICFP'15
Isabelle implementation:
ITP'14
Case study:
IJCAR'14,
Incremental coinduction:
FOSSACS'10
Non-free datatypes:
LICS'10, ICFP'11, CPP'13

Other people's work:
Isabelle/ZF codata (Paulson)
Containers
(Abbott, Altenkirch, Ghani)
Fibrations
(Hermida, Jacobs)
Flexible Corecursion
(Turi/Plotkin, Bartels, Jacobs,
Milius, Hinze, Atkey/McBride)
Flexible Coinduction
(Rot, Bonsangue, Rutten, Silva,
Roșu, Endrullis, Hendriks,
Hur/Dreyer/Vafeiadis)
Corecursion in MiniAgda, Agda
(Abel)