A Concrete Introduction to Abstract (Co)Inductive Datatypes

TABLEAUX 2015

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Middlesex University
School of Science and Technology
Foundations of Computing Group

Constructions and results

- Classical
- Obtained in the past 4 years in joint work with

Jasmin Blanchette

Isabelle

Dmitriy Traytel

Overview

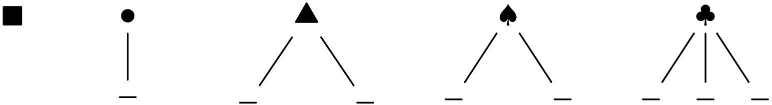
Part I: Datatypes

Part II: Codatatypes

Part I: Datatypes

Preliminariers: It’s All About Shape and Content

Shapes



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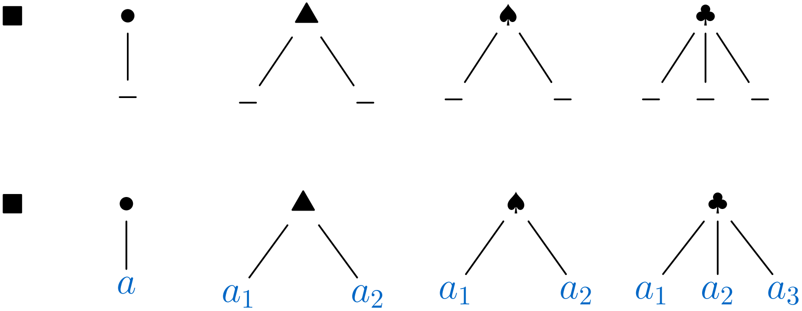

Shapes filled with content from a set $A = \{a_1, a_2, \ldots\}$

Natural Functors on Set

Set = the class of all sets

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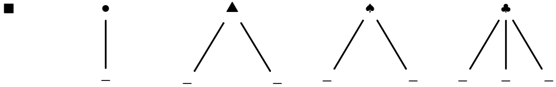

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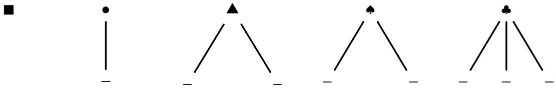

Each element $x \in \mathsf{F}\,A$ consists of:

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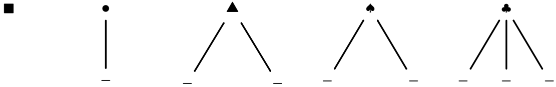

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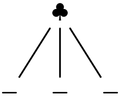

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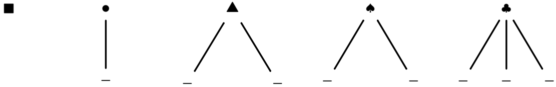

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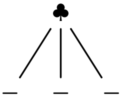

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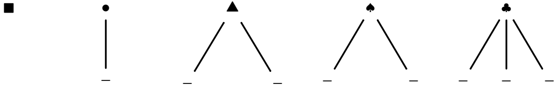

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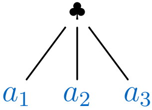

a filling with content from A, say

Examples of Natural Functors

$\mathsf{F}\ A = \mathbb{N} \times A$

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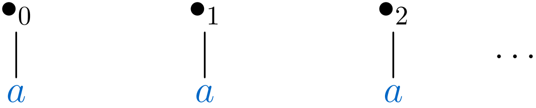

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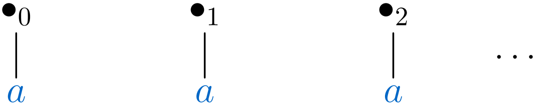

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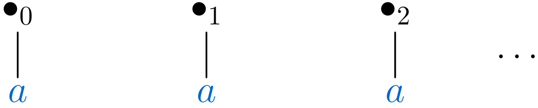

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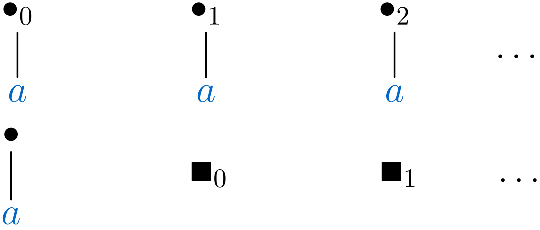

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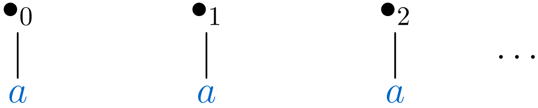

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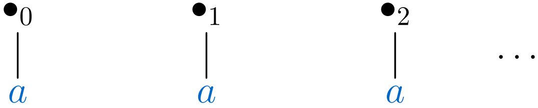

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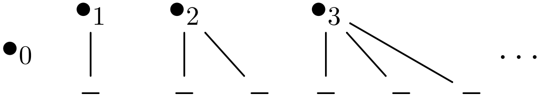

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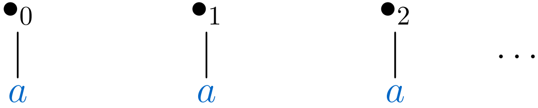

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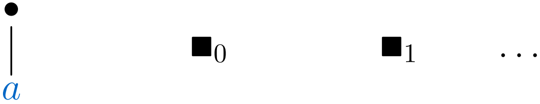

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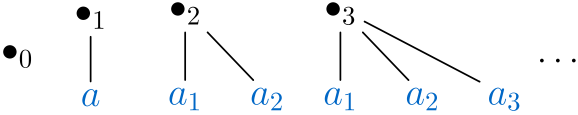

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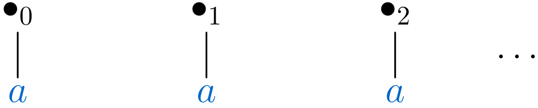

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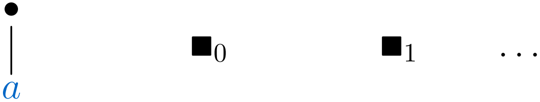

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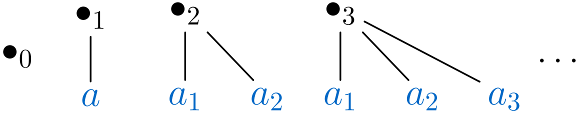

$\mathsf{F}\ A = \mathsf{Stream}\ A$?

Examples of Natural Functors

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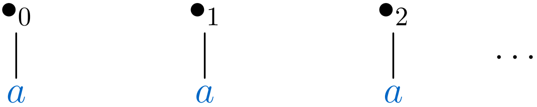

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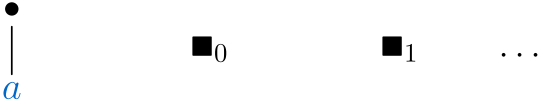

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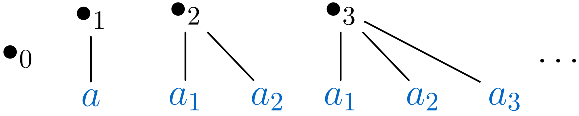

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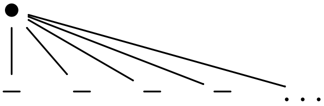

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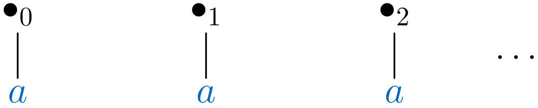
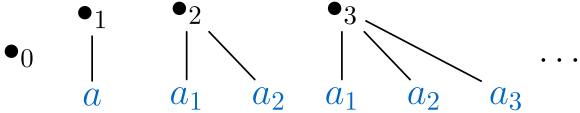
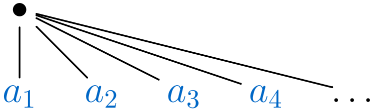


F $A = \mathbb{N} + A$

F $A =$ List A



F $A =$ Stream A



Examples of Natural Functors

F A = Lazy_List A ?

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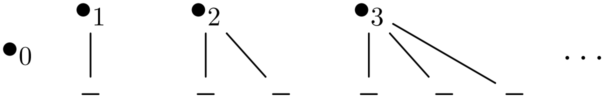
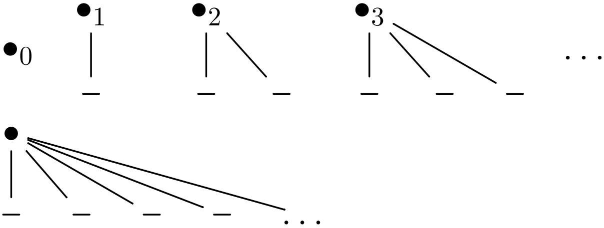


Examples of Natural Functors

F A = Lazy_List A = List A ∪ Stream A



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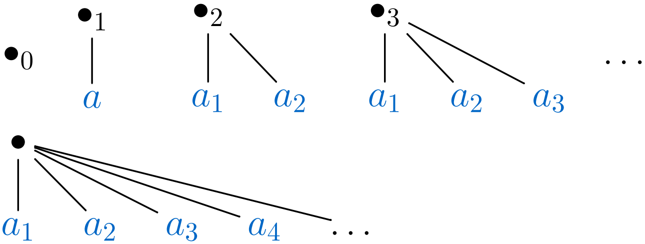

Examples of Natural Functors

$\mathsf{F}\ A = \mathsf{BTree}\ A$ (Full Binary Trees with leaves in A)

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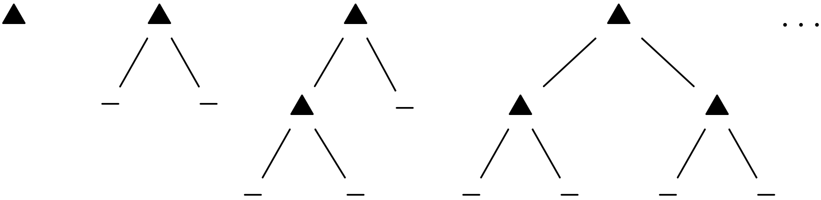

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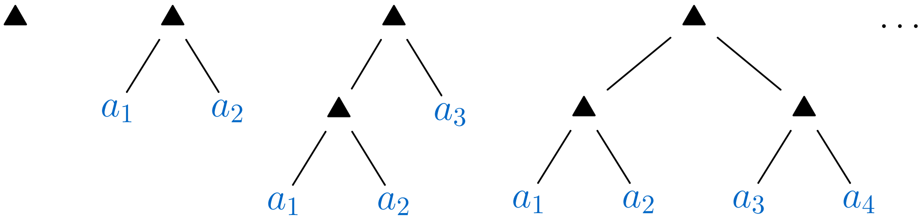

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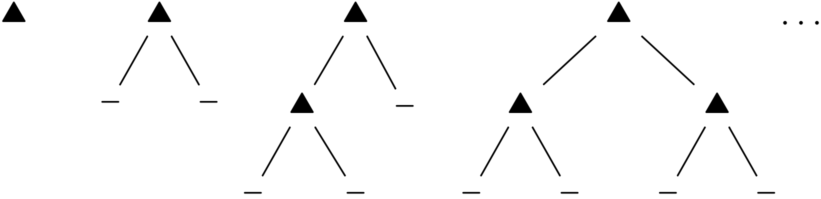

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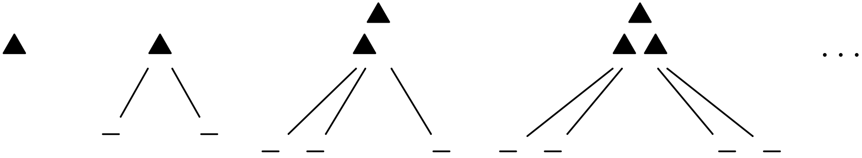

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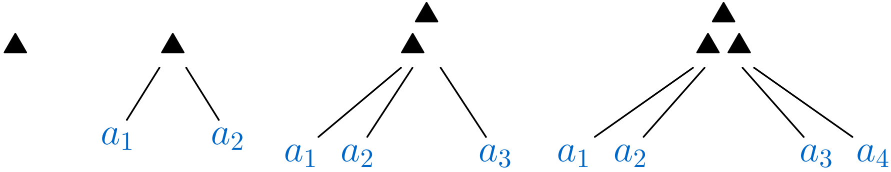

Functorial Action (Mapper)

$$\begin{tikzcd}
A \arrow[d, "f"] \\
B
\end{tikzcd}$$

Functorial Action (Mapper)

$$\begin{tikzcd}
A \arrow[d, "f"] & \mathsf{F}\,A \arrow[d, "\mathsf{F}\,f"] \\
B & \mathsf{F}\,B
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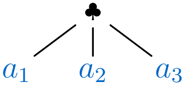

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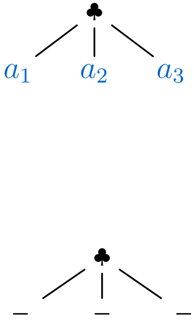

Keep the same shape

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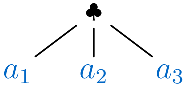
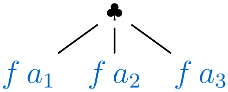


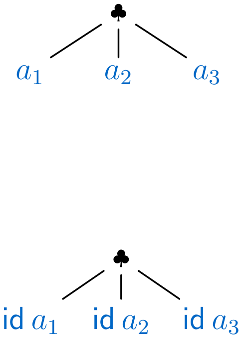


Keep the same shape

Apply f to the content

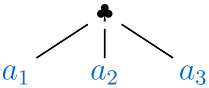
Commutation with the Identity Function

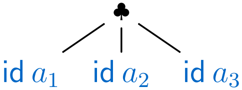
$$\begin{tikzcd}
A \arrow[d, "\mathsf{id}"] & \mathsf{F}\,A \arrow[d, "\mathsf{F}\,\mathsf{id}"] \\
A & \mathsf{F}\,A
\end{tikzcd}$$



Commutation with the Identity Function

$$\begin{tikzcd}
A \arrow[d, "\mathsf{id}"] & \mathsf{F}\,A \arrow[d, "\mathsf{F}\,\mathsf{id} = \mathsf{id}"] \\
A & \mathsf{F}\,A
\end{tikzcd}$$

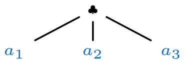


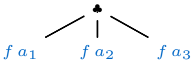


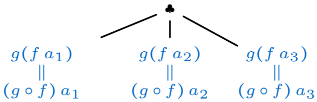
Commutation with Function Composition

$$\begin{tikzcd}
A \arrow[d, "f"] \\
B \arrow[d, "g"] \\
C
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{F}\,A \arrow[d, "\mathsf{F}\,f"] \arrow[dd, bend right, "\mathsf{F}\,(g \circ f)"'] \\
\mathsf{F}\,B \arrow[d, "\mathsf{F}\,g"] \\
\mathsf{F}\,C
\end{tikzcd}$$







Bottom Line

$\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

For all $A \xrightarrow{f} B$, we have $\mathsf{F}\,A \xrightarrow{\mathsf{F}\,f} \mathsf{F}\,B$ such that:

$$\mathsf{F}\,\mathsf{id}_A = \mathsf{id}_{\mathsf{F}\,A}$$
$$\mathsf{F}\,(g \circ f) = \mathsf{F}\,g \circ \mathsf{F}\,f$$

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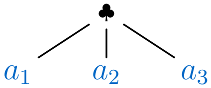
Functoriality

Atoms

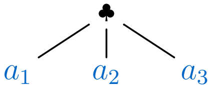
$$\begin{tikzcd}\mathsf{F}\ A \arrow[r, "\mathsf{Fatoms}_A"] & \mathcal{P}\ A\end{tikzcd}$$

Atoms



$$\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$$

Atoms



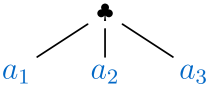
$\{a_1, a_2, a_3\}$

$$\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$$

Atoms

$$\begin{tikzcd}
\mathsf{F}\ A \arrow[r, "\mathsf{Fatoms}_A"] \arrow[d, "\mathsf{F}\ f"'] & \mathcal{P}\ A \arrow[d, "\mathsf{image}\ f"] \\
\mathsf{F}\ B \arrow[r, "\mathsf{Fatoms}_B"] & \mathcal{P}\ B
\end{tikzcd}$$

Atoms



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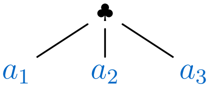
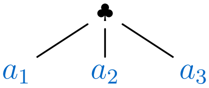


$\{a_1, a_2, a_3\}$

$$\begin{tikzcd}
\mathsf{F}\ A \arrow[r, "\mathsf{Fatoms}_A"] \arrow[d, "\mathsf{F}\ f"'] & \mathcal{P}\ A \arrow[d, "\mathsf{image}\ f"] \\
\mathsf{F}\ B \arrow[r, "\mathsf{Fatoms}_B"] & \mathcal{P}\ B
\end{tikzcd}$$

Atoms



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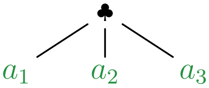
$\{f\ a_1, f\ a_2, f\ a_3\}$

Atoms



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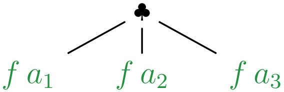

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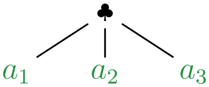

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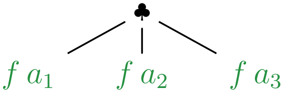

$\{f\ a_1, f\ a_2, f\ a_3\}$

Bottom Line

$\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

For all $A \xrightarrow{f} B$, we have $\mathsf{F}\,A \xrightarrow{\mathsf{F}\,f} \mathsf{F}\,B$ such that:

$$\mathsf{F}\ \mathsf{id}_A = \mathsf{id}_{\mathsf{F}A}$$
$$\mathsf{F}\,(g \circ f) = \mathsf{F}\,g \circ \mathsf{F}\,f$$

Functoriality

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Functoriality

For all A, we have $\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$ such that, for all $A \xrightarrow{f} B$:

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Functoriality

For all A, we have $\mathsf{F}\, A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\, A$ such that, for all $A \xrightarrow{f} B$:

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Naturality

Bottom Line: Natural Functors

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For all A, we have $\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$ such that, for all $A \xrightarrow{f} B$:

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Naturality

Examples of Natural Functors

$A \xrightarrow{f} B$

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Examples of Natural Functors

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$\mathsf{F}\ A = \mathbb{N} \times A$

Examples of Natural Functors

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Examples of Natural Functors

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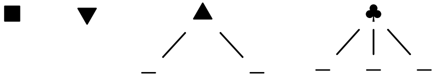
Iterating Shape Composition

Natural functor $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

Iterating Shape Composition

Natural functor $F : \mathsf{Set} \to \mathsf{Set}$

The shapes of F:



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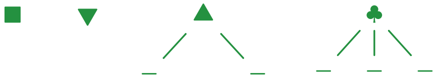
Natural functor $F : \mathsf{Set} \to \mathsf{Set}$

Copies of the shapes of F:



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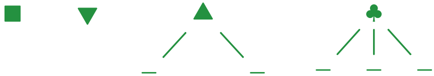

Put them together by plugging in shape for content slot

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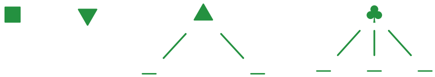

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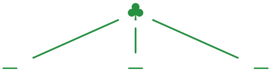

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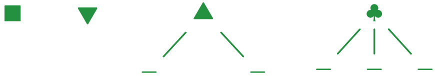

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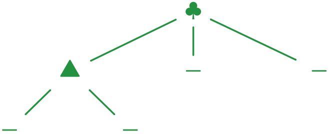

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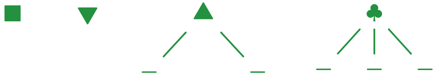

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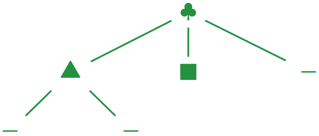

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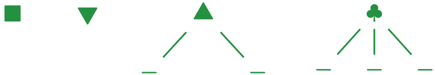

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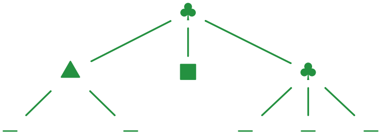

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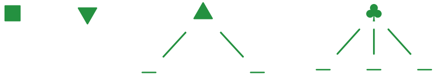
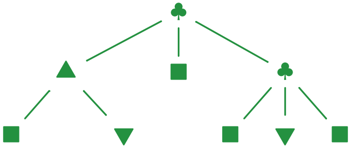


Put them together by plugging in shape for content slot until there are no lingering slots left!



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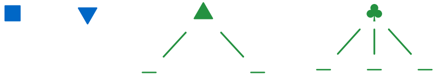

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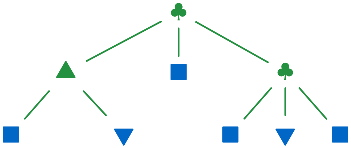

The leaves are always empty-content shapes

Iterating Shape Composition

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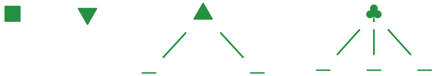

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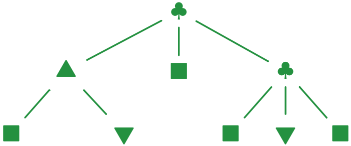
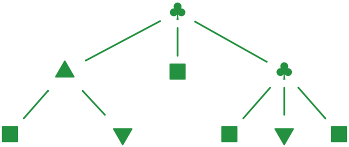


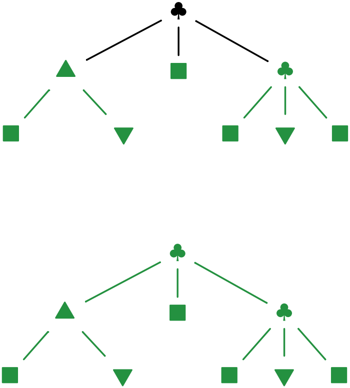
Define I_F = the set of all such finitary couplings

Properties of I_F: Bijectivity



$$\begin{tikzcd} F\, I_F \\ I_F \arrow[u, "\mathrm{dtor}"] \end{tikzcd}$$

Properties of I_F: Bijectivity



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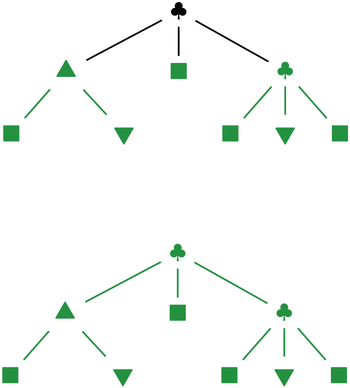

$$\begin{tikzcd}
F\,I_F \arrow[d, shift left, "\mathrm{ctor}"] \\
I_F \arrow[u, shift left, "\mathrm{dtor}"]
\end{tikzcd}$$

ctor and dtor are mutually inverse bijections

Properties of I_F: Iteration

$$\begin{tikzcd}
 & \mathsf{F}\,A \arrow[d, "s"'] \\
\mathsf{I}_\mathsf{F} & A
\end{tikzcd}$$

Properties of I_{F}: Iteration

$$\begin{tikzcd}
 & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_{\mathsf{F}} \arrow[r, dotted, "f"] & A
\end{tikzcd}$$

Properties of $\mathsf{I_F}$: Iteration

$$\begin{tikzcd}
 & \mathsf{F}\ A \arrow[d, "s"'] \\
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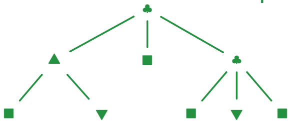
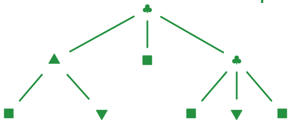
\end{tikzcd}$$

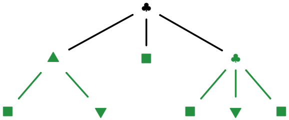


Properties of $\mathsf{I_F}$: Iteration

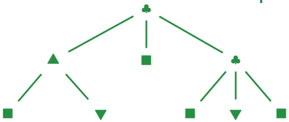
$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I_F} \arrow[u, "\mathsf{dtor}"'] \arrow[r, dotted, "f"] & A
\end{tikzcd}$$



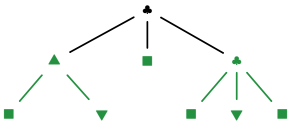
Properties of I_{F}: Iteration



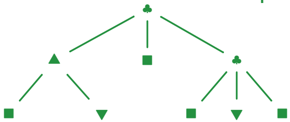
$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_{\mathsf{F}} & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_{\mathsf{F}} \arrow[u, "\mathsf{dtor}"'] \arrow[r, dotted, "f"] & A
\end{tikzcd}$$



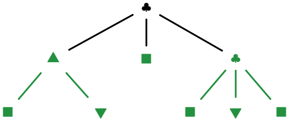
Properties of I_F: Iteration



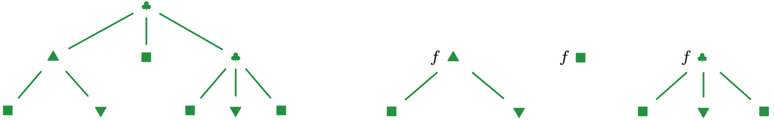
$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_\mathsf{F} \arrow[r, dotted, "\mathsf{F}\ f"'] & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_\mathsf{F} \arrow[u, "\mathsf{dtor}"'] \arrow[r, dotted, "f"] & A
\end{tikzcd}$$



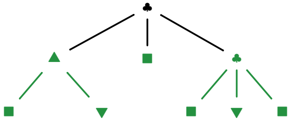
Properties of I_{F}: Iteration

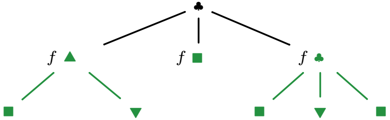


$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_{\mathsf{F}} \arrow[r, dotted, "\mathsf{F}\ f"'] & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_{\mathsf{F}} \arrow[u, "\mathsf{dtor}"'] \arrow[r, dotted, "f"] & A
\end{tikzcd}$$

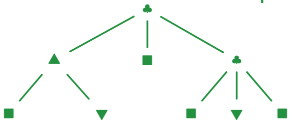


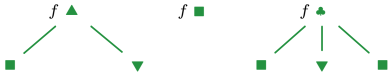
Properties of I_F: Iteration



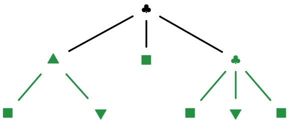


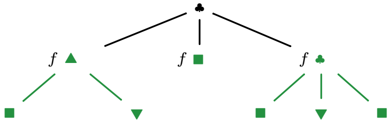
$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_\mathsf{F} \arrow[r, dotted, "\mathsf{F}\ f"'] & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_\mathsf{F} \arrow[u, "\mathsf{dtor}"'] \arrow[r, dotted, "f"] & A
\end{tikzcd}$$



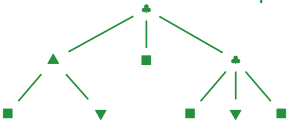


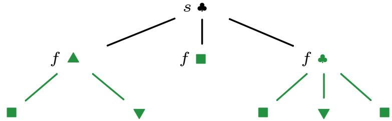
Properties of I_F: Iteration





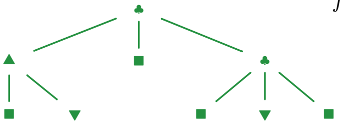
$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_\mathsf{F} \arrow[r, dotted, "\mathsf{F}\ f"'] & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_\mathsf{F} \arrow[u, "\mathsf{dtor}"'] \arrow[r, dotted, "f"] & A
\end{tikzcd}$$





Properties of $\mathsf{I_F}$: Iteration

$$\begin{tikzcd} & \mathsf{F}\ A \arrow[d, "s"'] \\ \mathsf{I_F} \arrow[r, dotted, "f"'] & A \end{tikzcd}$$



Properties of I_F: Iteration

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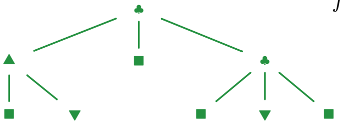
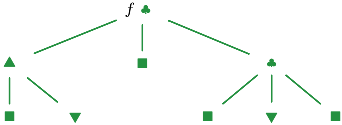
 & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_\mathsf{F} \arrow[r, dotted, "f"'] & A
\end{tikzcd}$$





Properties of I_F: Iteration

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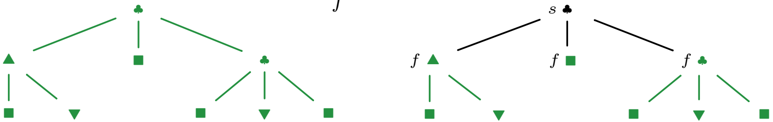
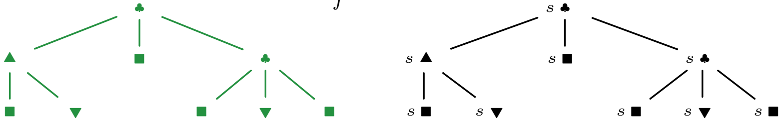
\mathsf{I}_\mathsf{F} \arrow[r, dotted, "f"'] & A
\end{tikzcd}$$



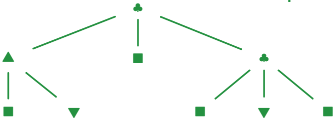
Properties of $\mathsf{I_F}$: Iteration

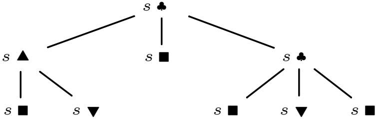
$$\begin{tikzcd} & \mathsf{F}\;A \arrow[d, "s"'] \\ \mathsf{I_F} \arrow[r, dotted, "f"'] & A \end{tikzcd}$$



Properties of I_{F}: Iteration

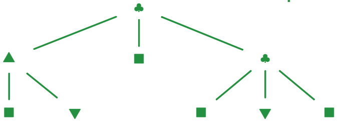
$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_{\mathsf{F}} \arrow[r, dotted, "\mathsf{F}\ f"] & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_{\mathsf{F}} \arrow[u, "\mathsf{dtor}"] \arrow[r, dotted, "f"'] & A
\end{tikzcd}$$

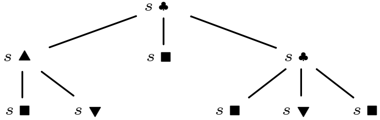




Properties of $\mathsf{I_F}$: Iteration

$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, dotted, "\mathsf{F}\ f"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I_F} \arrow[r, dotted, "f"'] & A
\end{tikzcd}$$





Properties of I_{F}: Iteration

$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_{\mathsf{F}} \arrow[r, dotted, "\mathsf{F}\ f"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"] \\
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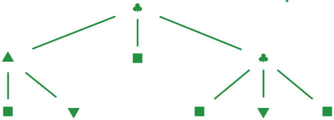
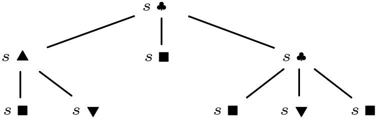
\end{tikzcd}$$

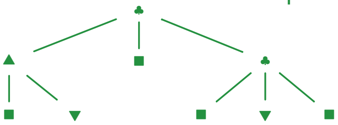


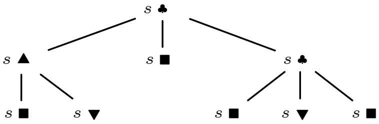


I_{F} is the initial F-algebra

Properties of I_{F}: Iteration

$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_{\mathsf{F}} \arrow[r, dotted, "\mathsf{F}\ f"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"'] \\
\mathsf{I}_{\mathsf{F}} \arrow[r, dotted, "f"'] & A
\end{tikzcd}$$





I_{F} is the initial F-algebra

$f = \mathsf{iter}_s$

Properties of I_F: Induction

I_F

Properties of I_F: Induction

I_F

φ unary predicate on I_F

Properties of I_F: Induction

I_F

φ unary predicate on I_F

Want: If

then $\forall i \in \mathsf{I}_\mathsf{F}.\ \varphi\ i$

Properties of $\mathsf{I_F}$: Induction

$$\begin{tikzcd} \mathsf{F}\ \mathsf{I_F} \\ \mathsf{I_F} \arrow[u, "\mathsf{dtor}"] \end{tikzcd}$$

φ unary predicate on $\mathsf{I_F}$
Want: If
then $\forall i \in \mathsf{I_F}.\ \varphi\ i$

Properties of $\mathsf{I_F}$: Induction

$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] & \mathcal{P}\ \mathsf{I_F} \\
\mathsf{I_F} \arrow[u, "\mathsf{dtor}"]
\end{tikzcd}$$

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Properties of I_F: Induction

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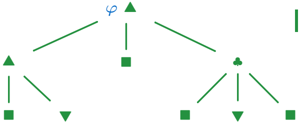
\mathsf{I}_\mathsf{F} \arrow[u, "\mathsf{dtor}"] &
\end{tikzcd}$$



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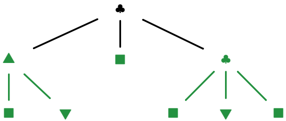
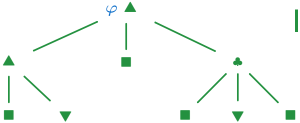
Want: If $\forall i \in \mathsf{I}_\mathsf{F}.$ $\Rightarrow \varphi\ i$

then $\forall i \in \mathsf{I}_\mathsf{F}.\ \varphi\ i$

Properties of $\mathsf{I_F}$: Induction



$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] & \mathcal{P}\ \mathsf{I_F} \\
\mathsf{I_F} \arrow[u, "\mathsf{dtor}"] &
\end{tikzcd}$$



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Want: If $\forall i \in \mathsf{I_F}.$ $\Rightarrow \varphi\ i$

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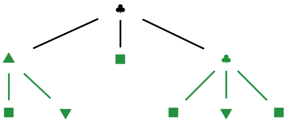
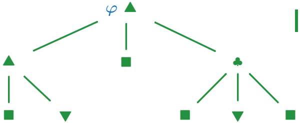
Properties of I_F: Induction



$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I}_\mathsf{F} \arrow[r, "\mathsf{Fatoms}"] & \mathcal{P}\ \mathsf{I}_\mathsf{F} \\
\mathsf{I}_\mathsf{F} \arrow[u, "\mathsf{dtor}"] &
\end{tikzcd}$$



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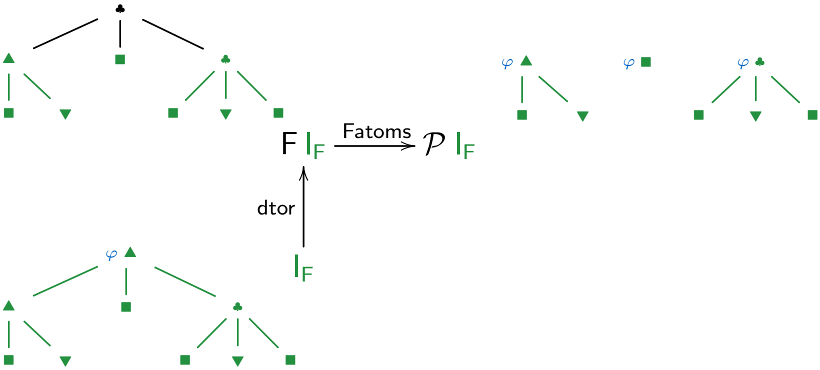

$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] & \mathcal{P}\ \mathsf{I_F} \\
\mathsf{I_F} \arrow[u, "\mathsf{dtor}"] &
\end{tikzcd}$$

φ unary predicate on $\mathsf{I_F}$

If $\forall i \in \mathsf{I_F}.\ (\forall i' \in \mathsf{Fatoms}\ (\mathsf{dtor}\ i).\ \varphi\ i') \Rightarrow \varphi\ i$

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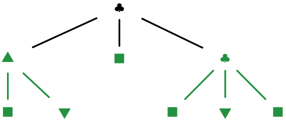
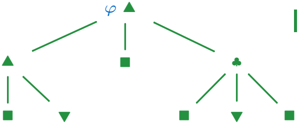
Properties of $\mathsf{I_F}$: Induction



$$\begin{tikzcd} \mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] & \mathcal{P}\ \mathsf{I_F} \\ \mathsf{I_F} \arrow[u, "\mathsf{dtor}"'] & \end{tikzcd}$$



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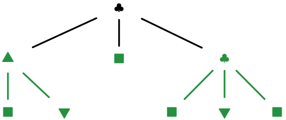
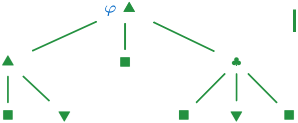
Properties of $\mathsf{I_F}$: Induction



$$\begin{tikzcd} \mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] & \mathcal{P}\ \mathsf{I_F} \\ \mathsf{I_F} \arrow[u, "\mathsf{dtor}"] \arrow[ur, "\mathsf{components}"'] & \end{tikzcd}$$



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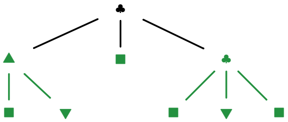
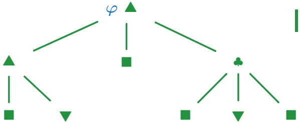
then $\forall i \in \mathsf{I_F}.\ \varphi\ i$

Properties of $\mathsf{I_F}$: Destructor-Style Induction



$$\begin{tikzcd}
\mathsf{F}\;\mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] & \mathcal{P}\;\mathsf{I_F} \\
\mathsf{I_F} \arrow[u, "\mathsf{dtor}"] \arrow[ur, "\mathsf{components}"'] &
\end{tikzcd}$$

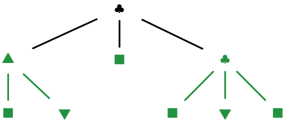


φ unary predicate on $\mathsf{I_F}$

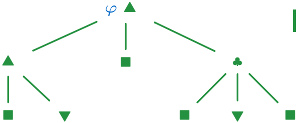
If $\forall i \in \mathsf{I_F}.\ (\forall i' \in \mathsf{components}\ i.\ \varphi\ i') \Rightarrow \varphi\ i$

then $\forall i \in \mathsf{I_F}.\ \varphi\ i$

Properties of $\mathsf{I_F}$: Constructor-Style Induction



$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] \arrow[d, "\mathsf{ctor}"'] & \mathcal{P}\ \mathsf{I_F} \\
\mathsf{I_F} &
\end{tikzcd}$$

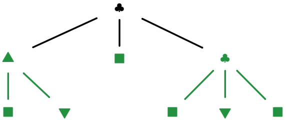


φ unary predicate on $\mathsf{I_F}$

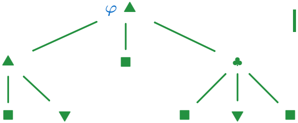
If $\forall i \in \mathsf{I_F}.\ (\forall i' \in \mathsf{components}\ i.\ \varphi\ i') \Rightarrow \varphi\ i$

then $\forall i \in \mathsf{I_F}.\ \varphi\ i$

Properties of $\mathsf{I_F}$: Constructor-Style Induction



$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{Fatoms}"] \arrow[d, "\mathsf{ctor}"'] & \mathcal{P}\ \mathsf{I_F} \\
\mathsf{I_F} &
\end{tikzcd}$$



φ unary predicate on $\mathsf{I_F}$

If $\forall x \in \mathsf{F}\ \mathsf{I_F}.\ (\forall i \in \mathsf{Fatoms}\ x.\ \varphi\ i) \Rightarrow \varphi\ (\mathsf{ctor}\ x)$
then $\forall i \in \mathsf{I_F}.\ \varphi\ i$

Bottom line for I_F

Given a natural functor F, $(I_F,\ \mathrm{ctor} : F\ I_F \to I_F)$ satisfies:

Bottom line for I_F

Given a natural functor F, $(I_F,\ \mathsf{ctor} : F\, I_F \to I_F)$ satisfies:

ctor bijection

Bottom line for $\mathsf{I_F}$

Given a natural functor F, $(\mathsf{I_F},\ \mathsf{ctor} : \mathsf{F}\ \mathsf{I_F} \to \mathsf{I_F})$ satisfies:

ctor bijection

Iteration (Initial Algebra Property): For all $(A, s : \mathsf{F}\ A \to A)$, there exists a unique function iter_s such that

$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{F}\ \mathsf{iter}_s"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"] \\
\mathsf{I_F} \arrow[r, "\mathsf{iter}_s"'] & A
\end{tikzcd}$$

Bottom line for $\mathsf{I_F}$

Given a natural functor F, $(\mathsf{I_F},\ \mathsf{ctor} : \mathsf{F\ I_F} \to \mathsf{I_F})$ satisfies:

ctor bijection

Iteration (Initial Algebra Property): For all $(A, s : \mathsf{F}\ A \to A)$, there exists a unique function iter_s such that

$$\begin{tikzcd}
\mathsf{F\ I_F} \arrow[r, "\mathsf{F\ iter}_s"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"] \\
\mathsf{I_F} \arrow[r, "\mathsf{iter}_s"'] & A
\end{tikzcd}$$

Induction: Given any predicate φ on $\mathsf{I_F}$

$$\frac{\forall x \in \mathsf{F\ I_F}.\ (\forall \mathsf{i} \in \mathsf{Fatoms\ x}.\ \varphi\ \mathsf{i}) \Rightarrow \varphi\ (\mathsf{ctor\ x})}{\forall i \in \mathsf{I_F}.\ \varphi\ \mathsf{i}}$$

Bottom line for $\mathsf{I_F}$

Given a natural functor F, $(\mathsf{I_F},\ \mathsf{ctor} : \mathsf{F}\ \mathsf{I_F} \to \mathsf{I_F})$ satisfies:

ctor bijection

$$\boxed{\mathsf{I_F} = \text{the datatype of F}}$$

Iteration (Initial Algebra Property): For all $(A, s : \mathsf{F}\ A \to A)$, there exists a unique function iter_s such that

$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "\mathsf{F}\ \mathsf{iter}_s"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"'] \\
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Example of Datatype

Let B be a fixed set. $\mathsf{F}\,A = \{*\} + B \times A$

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Let B be a fixed set. F $A = \{*\} + B \times A$

The shapes of F: Left $*$

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I.e., essentially lists $b_1 \cdot \ldots \cdot b_n$

So $\mathsf{I}_{\mathsf{F}} = \mathsf{List}_{\mathsf{B}}$

Example of Datatype: List

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$$\begin{tikzcd}
\mathsf{F}\ \mathsf{I_F} \arrow[r, "f"] \arrow[d, "\mathsf{ctor}"'] & \mathsf{F}\ A \arrow[d, "s"] \\
\mathsf{I_F} \arrow[r, "f"'] & A
\end{tikzcd}$$

$\forall x \in \mathsf{F}\ \mathsf{I_F}.\ \mathsf{f}\ (\mathsf{ctor}\ \mathsf{x}) = \mathsf{s}\ ((\mathsf{F}\ \mathsf{f})\ \mathsf{x})$

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\{*\} + B \times \mathsf{I_F} \arrow[r, "\{*\}+B\times f"] \arrow[d, "\mathsf{ctor}"'] & \{*\} + B \times A \arrow[d, "s"] \\
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Define:
$\mathsf{Nil} = \mathsf{ctor}\ (\mathsf{Left}\ *) \quad \mathsf{Cons}(b, i) = \mathsf{ctor}\ (\mathsf{Right}\ (b, i))$
$\mathsf{Nil}^A = s\ (\mathsf{Left}\ *) \quad \mathsf{Cons}^A(b, a) = s\ (\mathsf{Right}\ (b, a))$

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B \times \mathsf{I_F} \arrow[r, "B \times f"] \arrow[d, "\mathsf{Cons}"'] & B \times A \arrow[d, "\mathsf{Cons}^A"] \\
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$f\ \mathsf{Nil} = \mathsf{Nil}^A$

$\forall b \in B,\ i \in \mathsf{I_F}.\ \mathsf{f}\ (\mathsf{Cons}\ (\mathsf{b}, \mathsf{i})) = \mathsf{Cons}^{\mathsf{A}}\ (\mathsf{b}, \mathsf{f}\ \mathsf{i})$

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$f\ \mathsf{Nil} = \mathsf{Nil}^A$

We obtain standard list iteration!

$\forall b \in B,\ i \in \mathsf{I_F}.\ \mathsf{f}\ (\mathsf{Cons}\ (\mathsf{b}, \mathsf{i})) = \mathsf{Cons}^{\mathsf{A}}\ (\mathsf{b}, \mathsf{f}\ \mathsf{i})$

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Obtain standard list induction!

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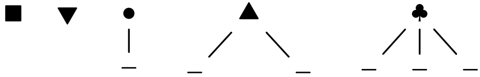
Iterating Shape Composition Revisited

Natural functor $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

Iterating Shape Composition Revisited

Natural functor F : Set → Set

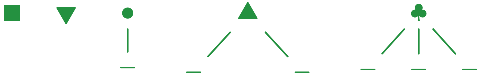
The shapes of F:



Iterating Shape Composition Revisited

Natural functor $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

Copies of the shapes of F:



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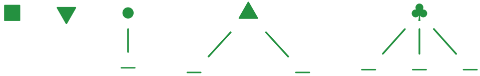

Put them together by plugging in shape for content slot

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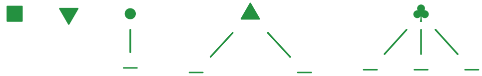

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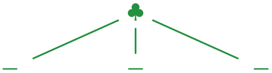

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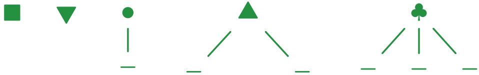

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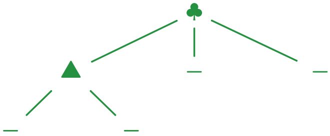

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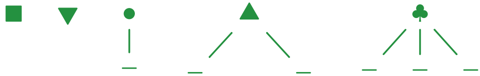

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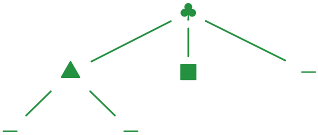

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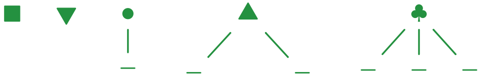

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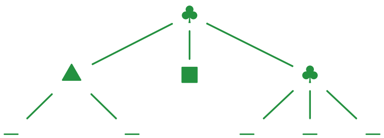

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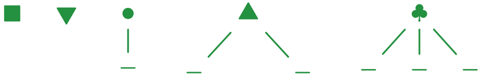
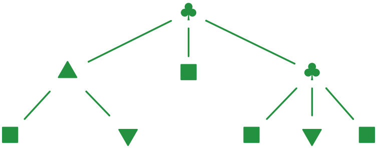


Put them together by plugging in shape for content slot until there are no lingering slots left!



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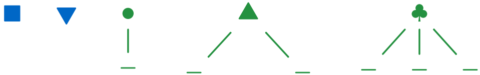

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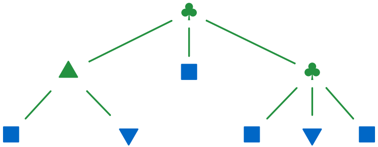
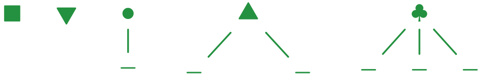


The leaves are always empty-content shapes

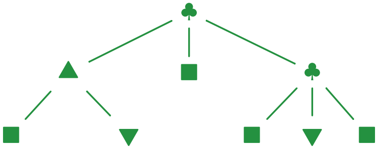
Iterating Shape Composition Revisited

Natural functor F : Set → Set

Copies of the shapes of F:



Put them together by plugging in shape for content slot until there are no lingering slots left!



The leaves are always empty-content shapes

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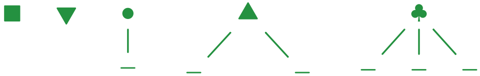

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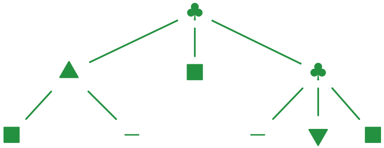

Allow infinite couplings

Iterating Shape Composition Revisited

Natural functor $F : \mathsf{Set} \to \mathsf{Set}$

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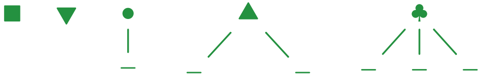

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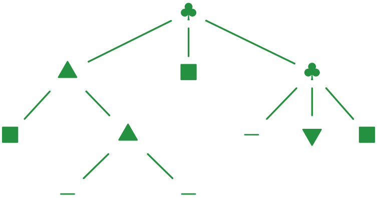

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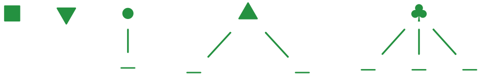

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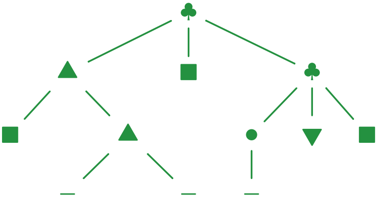

Allow infinite couplings

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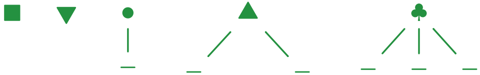

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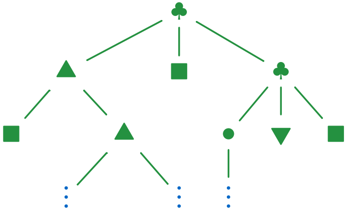

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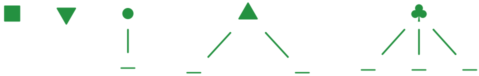

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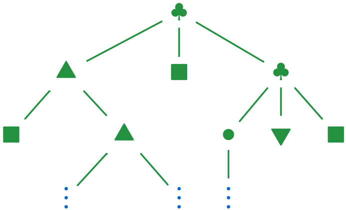
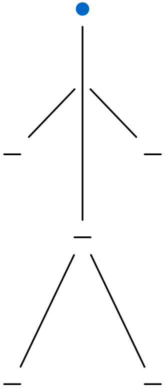


Define J_F = the set of all such (possibly) infinitary couplings

Welcome to Codatatypes

End of Part I



Many thanks for your attention
See you in 30 minutes