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Automated Reasoning Building Blocks

Part I

Christoph Weidenbach

Max Planck Institute for Informatics

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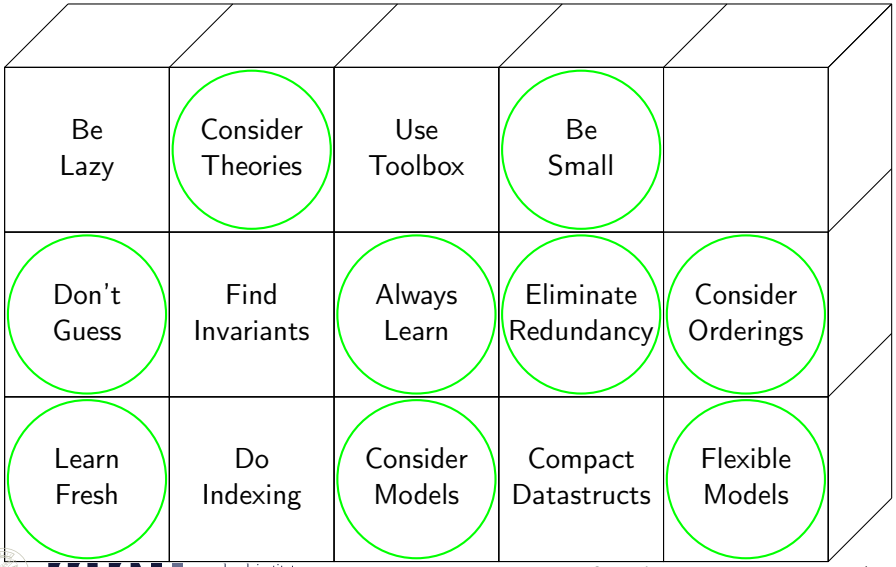
Outline

Automated Reasoning Building Blocks



Propositional Clause Logic

Syntax

- Clauses have the form $P \vee \neg R \vee Q \vee Q$ where $P, Q, R \in \Sigma$
- Clauses denoted by C, D, empty clause denoted by $\bot$
- Clause sets N, M are interpreted as conjunctions of clauses

Semantics

- (Partial) Valuations $\mathcal{A} : \Sigma \to \{0, 1\}$
- Clause Set N satisfiable if $\mathcal{A}(N) = 1$ for some $\mathcal{A}$, $\mathcal{A} \models N$
- Clause Set N unsatisfiable if $\mathcal{A}(N) = 0$ for all $\mathcal{A}$
- Clause Set N valid if $\mathcal{A}(N) = 1$ for all $\mathcal{A}$, $\models N$

$N = \{P \vee Q,\ \neg P \vee Q,\ P \vee \neg Q,\ \neg P \vee \neg Q\}$

Analytic Models

Beth 55, Smullyan 68, Fitting 90

Tableau Procedure: Close Branch & Backtrack, Split Disjunction

$N = \{P \vee Q,\ R \vee S,\ \neg P \vee Q,\ \neg Q\}$

$\Rightarrow_{\mathsf{TAB}}\ ([], \top)$
$\Rightarrow_{\mathsf{TAB}}\ ([P^{P \vee Q}], \top)$
$\Rightarrow_{\mathsf{TAB}}\ ([P^{P \vee Q}\, R^{R \vee S}], \top)$
$\Rightarrow_{\mathsf{TAB}}\ ([P^{P \vee Q}\, R^{R \vee S}\, \neg P^{\neg P \vee Q}], \bot)$
$\Rightarrow_{\mathsf{TAB}}\ ([P^{P \vee Q}\, R^{R \vee S}\, Q], \top)$
$\Rightarrow_{\mathsf{TAB}}\ ([P^{P \vee Q}\, R^{R \vee S}\, Q\, \neg Q], \bot)$
$\Rightarrow_{\mathsf{TAB}}\ ([P^{P \vee Q}\, S], \top) \ldots$

$\Rightarrow_{\mathsf{TAB}}\ ([Q], \top)$
$\Rightarrow_{\mathsf{TAB}}\ ([Q\, \neg Q], \bot)$
$\Rightarrow_{\mathsf{TAB}}\ ([], \bot)$

Only Guessing &
No Model
Consideration

Consider Models

Davis & Logman & Loveland 1962

DPLL Procedure: Find Conflict & Backtrack, Propagate, Guess

$N = \{P \vee Q,\ R \vee S,\ \neg P \vee Q,\ \neg Q\}$

$\Rightarrow_{\mathsf{DPLL}}\ ([], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([\neg Q^{\neg Q}], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([\neg Q^{\neg Q} P^{P \vee Q}], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([\neg Q^{\neg Q} P^{P \vee Q}], \neg P \vee Q)$
$\Rightarrow_{\mathsf{DPLL}}\ ([], \bot)$

Propagation &
No Guessing

DPLL Minimal Proof Length

$N = \{P \vee Q,\ \neg P \vee Q,\ P \vee \neg Q,\ \neg P \vee \neg Q\}$

$\Rightarrow_{\mathsf{DPLL}}\ ([], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([P^1], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([P^1 Q^{\neg P \vee Q}], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([P^1 Q^{\neg P \vee Q}], \neg P \vee \neg Q)$
$\Rightarrow_{\mathsf{DPLL}}\ ([\neg P], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([\neg P, Q^{P \vee Q}], \top)$
$\Rightarrow_{\mathsf{DPLL}}\ ([\neg P, Q^{P \vee Q}], P \vee \neg Q)$
$\Rightarrow_{\mathsf{DPLL}}\ ([], \bot)$

No Learning
$O(2^n)$

Always Learn

Robinson 1965

Resolution: $C \vee P,\ \neg P \vee D \Rightarrow_{\mathsf{RES}} C \vee D$

$N = \{P \vee Q,\ \neg P \vee Q,\ P \vee \neg Q,\ \neg P \vee \neg Q\}$

$\neg P \vee Q,\ \neg P \vee \neg Q \Rightarrow_{\mathsf{RES}} \neg P$
$P \vee Q,\ P \vee \neg Q \Rightarrow_{\mathsf{RES}} P$
$P,\ \neg P \Rightarrow_{\mathsf{RES}} \bot$

$O(n)$
No Model
Consideration

Eliminate Redundancy

$N = \{P \vee Q,\ \neg P \vee \neg Q\}$
$P \vee Q,\ \neg P \vee \neg Q \Rightarrow_{\mathrm{RES}} Q \vee \neg Q$
May we eliminate $Q \vee \neg Q$?

$N = \{P \vee Q \vee R,\ \neg P \vee Q, P \vee R\}$
$\neg P \vee Q,\ P \vee R \Rightarrow_{\mathrm{RES}} Q \vee R$
May we eliminate $P \vee Q \vee R$?

Number of clauses generated in a typical resolution run:
42, 420, 42000, 42000000, ...

Redundancy: A First-Class Citizen

Boyer 1971

Lock Resolution: $C \vee P_k, \neg P_l \vee D \Rightarrow_{\mathrm{LRES}} C \vee D$
if k, l are maximal indexes, respectively

$N = \{P_5 \vee Q_4,\ \neg P_4 \vee Q_5,\ P_4 \vee \neg Q_5,\ \neg P_5 \vee \neg Q_4\}$
$\Rightarrow_{\mathrm{LRES}} Q_4 \vee \neg Q_4$
$\Rightarrow_{\mathrm{LRES}} \neg P_4 \vee P_4$

Redundancy needs to be considered in the context of a calculus.

Learn Fresh, Eliminate Redundancy

Bachmair & Ganzinger 1990

Superposition: $C \vee P, \neg P \vee D \Rightarrow_{\mathrm{LRES}} C \vee D$
if P, $\neg P$ are maximal in their respective clauses

Ordering

- $\prec$ is a total strict ordering on Σ: $P \prec Q$
- $\prec$ on literals: $P \prec \neg P \prec Q \prec \neg Q$
- $\prec$ on clauses: multiset extension: $\{P, Q\} \prec \{P, Q, Q\} \prec \{\neg Q\}$
- $N^{\prec C} = \{D \in N \mid D \prec C\}$

Superposition Redundancy

Definition (Redundancy)

A clause C is *redundant* with respect to a clause set N if $N^{\prec C} \models C$.

$P \prec Q \prec R$
$N = \{P \vee Q,\ R \vee \neg P\}$ clauses $P \vee Q \vee \neg R,\ Q \vee R$ redundant

Superposition Static Model $N_{\mathcal{I}}$

$$
\begin{aligned}
N_C &:= \textstyle\bigcup_{D \prec C} \delta_D \\
\delta_D &:= \begin{cases} \{P\} & \text{if } D = D' \vee P, P \text{ strictly maximal } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases} \\
N_{\mathcal{I}} &:= \textstyle\bigcup_{C \in N} \delta_C
\end{aligned}
$$

Superposition Results

Theorem (Completeness, Models, Redundancy)

If all superposition inferences in N up to redundancy are performed and $\bot \notin N$ then N is satisfiable and $N_{\mathcal{I}} \models N$.
It is sufficient to consider inferences between a minimal false clause $\neg P \vee C$, $N_{\mathcal{I}} \not\models \neg P \vee C$ and its productive counterpart $P \vee D$.
The result $C \vee D$ of the superposition inference is not redundant.

Model Properties

- fixed by ordering: $P \vee Q$, $P \vee \neg Q$
- minimal: $P \vee Q$, $\neg P \vee R$, where $R \prec Q \prec P$ then $N_{\mathcal{I}} = \{P\}$

Static Ordering & Model

Always Learn Fresh, Flexible Models

Silva, Sakallah, Bayardo, Schrag, Et Al 2000-

CDCL: Find Conflict & Backtrack & Learn, Propagate, Guess

$N = \{P \vee Q,\ \neg P \vee Q,\ P \vee \neg Q,\ \neg P \vee \neg Q\}$

$\Rightarrow_{\text{CDCL}}$ $([], \emptyset, \top)$
$\Rightarrow_{\text{CDCL}}$ $([P^1], \emptyset, \top)$
$\Rightarrow_{\text{CDCL}}$ $([P^1 Q^{\neg P \vee Q}], \emptyset, \top)$
$\Rightarrow_{\text{CDCL}}$ $([P^1 Q^{\neg P \vee Q}], \emptyset, \neg P \vee \neg Q)$
$\Rightarrow_{\text{CDCL}}$ $([\neg P], \{\neg P\}, \top)$
$\Rightarrow_{\text{CDCL}}$ $([\neg P, Q^{P \vee Q}], \{\neg P\}, \top)$
$\Rightarrow_{\text{CDCL}}$ $([\neg P, Q^{P \vee Q}], \{\neg P\}, P \vee \neg Q)$
$\Rightarrow_{\text{CDCL}}$ $([], \{\neg P, \bot\}, \bot)$

$O(n)$
No Redundancy

Consider Theories

while Program Analysis

1	$n = 0$;	$\neg P_1(n,x,y) \vee P_2(0,x,y)$
2	while $(x > 0)$ {	$\neg x > 0 \vee \neg P_2(n,x,y) \vee P_3(n,x,y)$
		$\neg x \leq 0 \vee \neg P_2(n,x,y) \vee P_6(n,x,y)$
3	$n = n + y$;	$\neg P_3(n,x,y) \vee P_4(n+y,x,y)$
4	$x = x - 1$;	$\neg P_4(n,x,y) \vee P_5(n,x-1,y)$
5	}	$\neg P_5(n,x,y) \vee P_2(n,x,y)$
6	return n;	

Axiomatize Theories

Program Formalization

$\neg x > 0 \vee \neg P_2(n,x,y) \vee P_3(n,x,y)$
$\neg x \leq 0 \vee \neg P_2(n,x,y) \vee P_6(n,x,y)$
$\neg P_3(n,x,y) \vee P_4(n+y,x,y)$
$\neg P_4(n,x,y) \vee P_5(n,x-1,y)$

Theory Axioms

$x = 0 \vee x > 0 \wedge s(x) > x$
$s(x) + y = s(x+y) \wedge 0 + y = y \wedge x + y = y + x$
$\ldots$

Incomplete & No Decision Procedure

Combine Theories

Program Formalization

$\neg x > 0 \vee \neg P_2(n,x,y) \vee P_3(n,x,y)$
$\neg x \leq 0 \vee \neg P_2(n,x,y) \vee P_6(n,x,y)$
$\neg P_3(n,x,y) \vee P_4(n+y,x,y)$
$\neg P_4(n,x,y) \vee P_5(n,x-1,y)$

Combination

$x > 0 \quad \| \neg P_2(n,x,y) \vee P_3(n,x,y)$
$x \leq 0 \quad \| \neg P_2(n,x,y) \vee P_6(n,x,y)$
$z = n+y \| \neg P_3(n,x,y) \vee P_4(z,x,y)$
$z = x-1 \| \neg P_4(n,x,y) \vee P_5(n,z,y)$

Expressive Logic & Difficult Automation

Summary

Open Questions

- Can redundancy and completeness be combined?
- Can redundancy and ordering restrictions be combined?
- Can model building and inferences be combined?
- How does reasoning in combination of theories work?

Answers

- Superposition [BachmairGanzinger90]
- CDCL [BayardoSchrag96, SilvaSakallah96, NieuwenhuisEtAl06]
- CDCL(T)/SMT [NieuwenhuisEtAl06]
- Hierarchic Superposition [BachmairEtAl94, KruglovW12, FietzkeW12]

Disclaimer

Simplified Presentation

- Many Technical Details Omitted
- Almost no Equality
- See References for Further Reading

Goal

Get the Ideas/Intuition

References Tableau

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Propositional Resolution

Resolution [Robinson65]

Resolution
$(N \uplus \{C \vee P, D \vee \neg P\}) \Rightarrow_{\text{RES}} (N \cup \{C \vee P, D \vee \neg P, C \vee D\})$

Factoring
$(N \uplus \{C \vee L \vee L\}) \Rightarrow_{\text{RES}} (N \cup \{C \vee L \vee L\} \cup \{C \vee L\})$

Theorem (Resolution is Sound and Complete)

N is unsatisfiable iff $N \Rightarrow^*_{RES} N' \cup \{\bot\}$

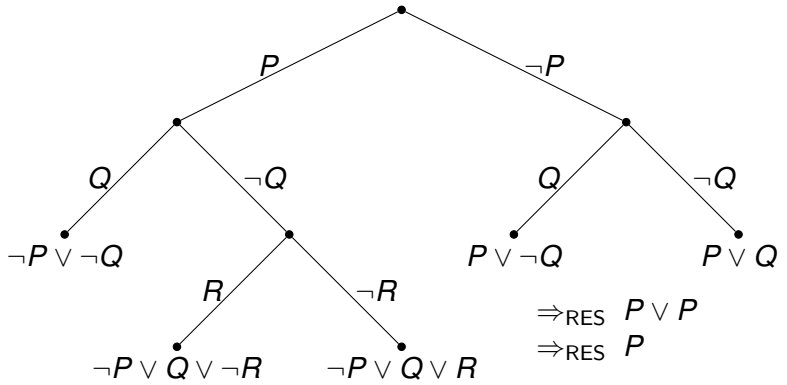
Soundness

Resolution
$(N \uplus \{C \vee P, D \vee \neg P\}) \Rightarrow_{\text{RES}} (N \cup \{C \vee P, D \vee \neg P, C \vee D\})$
if $\mathcal{A}((C \vee P) \wedge (D \vee \neg P)) = 1$ then $\mathcal{A}(C \vee D) = 1$

Factoring
$(N \uplus \{C \vee L \vee L\}) \Rightarrow_{\text{RES}} (N \cup \{C \vee L \vee L\} \cup \{C \vee L\})$
if $\mathcal{A}(C \vee L \vee L) = 1$ then $\mathcal{A}(C \vee L) = 1$

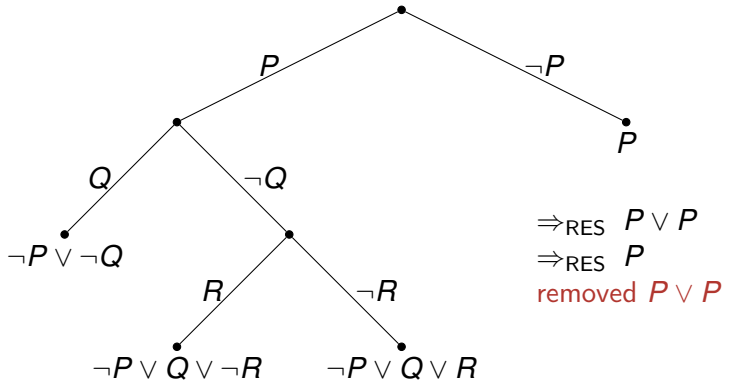
Completeness: Semantic Trees

$N = \{P \vee Q,\ \neg P \vee \neg Q,\ P \vee \neg Q,\ \neg P \vee Q \vee R,\ \neg P \vee Q \vee \neg R\}$



Semantic Tree Redundancy

$N = \{P \vee Q,\ \neg P \vee \neg Q,\ P \vee \neg Q,\ \neg P \vee Q \vee R,\ \neg P \vee Q \vee \neg R,\ P\}$



Eliminate Redundancy

Well-Founded Semantic Tree Ordering
A clause C is redundant if $D \subseteq C$ for some D
Any clause C not occurring in any semantic tree is redundant

Subsumption
$(N \uplus \{C, D\}) \Rightarrow_{\text{RES}} (N \cup \{C\})$
provided $C \subset D$

Condensation
$(N \uplus \{C \vee L \vee L\}) \Rightarrow_{\text{RES}} (N \cup \{C \vee L\})$

Tautology Deletion
$(N \uplus \{C \vee P \vee \neg P\}) \Rightarrow_{\text{RES}} (N)$

Can redundancy and completeness be combined?

Termination

$N_0 \Rightarrow_{\mathsf{RES}} N_1 \Rightarrow_{\mathsf{RES}} \ldots$

How Many Different Clause Sets?

at most 3^n different clauses, $n = |\Sigma|$, modulo condensation
at most $2^{(3^n)}$ different clause sets

Examples

$N = \{P \vee C_1, P \vee C_2, \ldots\}$ obviously satisfiable
$N = \{P \vee Q,\ P \vee \neg Q\}$
under $\mathcal{A}(P) = 1$ why consider $P \vee \neg Q$ for resolution?

Can model building and inferences be combined?

Propositional Superposition

Ordering

- $\prec$ is a total strict ordering on Σ: $P \prec Q$
- $\prec$ on literals: $P \prec \neg P \prec Q \prec \neg Q$
- $\prec$ on clauses: multiset extension: $\{P, Q\} \prec \{P, Q, Q\} \prec \{\neg Q\}$
- $N^{\prec C} = \{D \in N \mid D \prec C\}$

Definition (Redundancy)

A clause C is *redundant* with respect to a clause set N if $N^{\prec C} \models C$.

$P \prec Q \prec R$
$N = \{P \vee Q,\ R \vee \neg P\}$ clauses $P \vee Q \vee \neg R,\ Q \vee R$ redundant

Models

Definition (Selection Function)

- sel maps clauses to one of its negative literals or $\bot$
- if $\mathsf{sel}(C) = \neg P$ then $\neg P$ is called *selected* in C
- if $\mathsf{sel}(C) = \bot$ then no literal in C is *selected*

Partial Herbrand Model Construction

$$N_C := \bigcup_{D \prec C} \delta_D$$

$$\delta_D := \begin{cases} \{P\} & \text{if } D = D' \vee P, P \text{ strictly maximal, no literal selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

$$N_{\mathcal{I}} := \bigcup_{C \in N} \delta_C$$

Partial Herbrand Model Properties

$$N_C := \bigcup_{D \prec C} \delta_D$$

$$\delta_D := \begin{cases} \{P\} & \text{if } D = D' \vee P, P \text{ strictly maximal, no literal selected in } D \text{ and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

$$N_{\mathcal{I}} := \bigcup_{C \in N} \delta_C$$

Properties

- $N_{\mathcal{I}}$ is minimal with respect to set inclusion
- if $D \prec C$ and $N_C \models D$ then $N_{\mathcal{I}} \models D$
- if $P \vee P \prec C$ then $\delta_C \neq \{P\}$
- if $\delta_C = \{P\}$ then $N_C \cup \delta_C \models C$

Propositional Superposition

Superposition

Superposition Left

$(N \uplus \{C \vee P, D \vee \neg P\}) \Rightarrow_{\mathsf{SUP}} (N \cup \{C \vee P, D \vee \neg P\} \cup \{C \vee D\})$

where (i) P is strictly maximal in $C \vee P$
(ii) no literal in $C \vee P$ is selected
(iii) $\neg P$ is maximal and no literal selected in $D \vee \neg P$, or
$\neg P$ is selected in $D \vee \neg P$

Factoring

$(N \uplus \{C \vee P \vee P\}) \Rightarrow_{\mathsf{SUP}} (N \cup \{C \vee P \vee P\} \cup \{C \vee P\})$

where (i) P is maximal in $C \vee P \vee P$
(ii) no literal is selected in $C \vee P \vee P$

Saturation and Completeness

Definition (Saturation)

A set N of clauses is called *saturated up to redundancy*, if any clause generated by Superposition Left or Factoring from non-redundant clauses in N is redundant with respect to N or contained in N.

Theorem (Superposition Completeness)

If N is saturated up to redundancy and $\bot \notin N$ then N is satisfiable and $N_{\mathcal{I}} \models N$.

Superposition Completeness Proof

Theorem (Superposition Completeness)

If N is saturated up to redundancy and $\bot \notin N$ then N is satisfiable and $N_{\mathcal{I}} \models N$.

Proof.

By contradiction. I assume:
(i) if $N \Rightarrow_{\mathsf{SUP}} N \cup D$ then $N^{\prec D} \models D$ or $D \in N$, (ii) $\bot \notin N$ and
(iii) $N_{\mathcal{I}} \not\models N$.
Then there is a minimal clause $C \vee L \in N$ such that $N_{\mathcal{I}} \not\models C \vee L$ and L is selected or nothing selected and L maximal. This clause must exist because $\bot \notin N$.
$C \vee L$ is not redundant and by Superposition Left or Factoring we can derive a non-redundant clause that is smaller than $C \vee L$ and false in $N_{\mathcal{I}}$, a contradiction. □

Proof: More Details.

- if $C \vee L$ redundant, then $N_{C \vee L} \models C \vee L$, a contradiction
- L positive, L not strictly maximal, do Factoring, a contradiction
- L positive, L strictly maximal, $L \in N_{\mathcal{I}}$, a contradiction
- L negative, do Superposition Left, a contradiction

□

Completeness Summary

Superposition Completeness

- does inferences with respect to a candidate model $N_{\mathcal{I}}$
- only inferences on false clauses needed: see proof
- supports ordering restrictions: $\prec$
- supports redundancy: $N^C \models C$

Superposition Results

Theorem (Completeness, Models, Redundancy)

If all superposition inferences in N up to redundancy are performed and $\bot \notin N$ then N is satisfiable and $N_{\mathcal{I}} \models N$.
It is sufficient to consider inferences between a minimal false clause $\neg P \vee C$, $N_{\mathcal{I}} \not\models \neg P \vee C$ and its productive counterpart $P \vee D$.
The result $C \vee D$ of the superposition inference is not redundant.

Propositional Specialities

- $N_{\mathcal{I}}$ can be effectively constructed
- $N_{\mathcal{I}} \not\models \neg P \vee C$ easy to decide
- unsatisfiability is co-NP, saturation always terminates

Concrete Redundancy

Subsumption
$(N \uplus \{C, D\}) \Rightarrow_{\mathsf{SUP}} (N \cup \{C\})$
provided $C \subset D$

Condensation
$(N \uplus \{C \vee L \vee L\}) \Rightarrow_{\mathsf{SUP}} (N \cup \{C \vee L\})$

Tautology Deletion
$(N \uplus \{C \vee P \vee \neg P\}) \Rightarrow_{\mathsf{SUP}} (N)$

Subsumption Resolution
$(N \uplus \{C_1 \vee L, C_2 \vee \neg L\}) \Rightarrow_{\mathsf{SUP}} (N \cup \{C_1 \vee L, C_2\})$
where $C_1 \subseteq C_2$

Extension I: Saturation

Example (N saturated vs. $N_{\mathcal{I}} \models N$)

$N = \{P, \neg P \vee \neg Q\}$ with ordering $Q \prec P$.
Then $N_{\mathcal{I}} \models N$ but N is not saturated.

Fix

Always check whether $N_{\mathcal{I}} \models N$.

Extension II: Flexible Model Operator

Example (Inflexible Model Operator)

$N = \{P \vee Q,\ \neg P \vee R\}$ with ordering $R \prec Q \prec P$.
Then $N_{\mathcal{I}} = \{P\}$ and $N_{\mathcal{I}} \not\models N$
But $N_{\mathcal{I}} \cup \{R\} \models N$.

Fix

Use a different model operator.

Flexible Model Operator

H is a decision heuristic: $H\colon \Sigma \to \{0, 1\}$

$$N_P^H := \bigcup_{Q \prec P} \delta_Q^H$$

$$\delta_P^H := \begin{cases} \{P\} & \text{if } (D \vee P) \in N, \text{ with } N_P^H \models \neg D \\ & \text{and } P \text{ strictly maximal, nothing selected} \quad \text{or} \\ & H(P) = 1 \text{ no clause } (D' \vee \neg P) \in N, D' \prec P \\ & \text{such that } N_P^H \models \neg D' \\ \emptyset & \text{otherwise} \end{cases}$$

$$N_\Sigma^H := \bigcup_{P \in \Sigma} \delta_P^H$$

N_Σ^H Properties [W2015]

Theorem (Superposition Completeness)

If N is saturated up to redundancy and $\bot \notin N$ then N is satisfiable and $N_\Sigma^H \models N$.

Example (Flexible Model Operator)

$N = \{P \vee Q,\ \neg P \vee R\}$ with ordering $R \prec Q \prec P$, $H(R) = 1$
Then $N_\Sigma^H = \{P, R\}$ and $N_\Sigma^H \models N$

CDCL States

$(\epsilon; N; \emptyset; 0; \top)$	is the start state for some clause set N
$(M; N; U; k; \top)$	is a final state, if $M \models N$ and all literals from N are defined in M
$(M; N; U; k; \bot)$	is a final state, where N has no model
$(M; N; U; k; \top)$	is an intermediate model search state if $M \not\models N$ or not all literals from N are defined in M
$(M; N; U; k; D)$	is a backtracking state if $D \notin \{\top, \bot\}$

CDCL Rules I

Model Extension Rules

Propagate
$(M; N; U; k; \top) \Rightarrow_{\text{CDCL}} (ML^{C \vee L}; N; U; k; \top)$
provided $C \vee L \in (N \cup U)$, $M \models \neg C$, and L is undefined in M

Decide
$(M; N; U; k; \top) \Rightarrow_{\text{CDCL}} (ML^{k+1}; N; U; k+1; \top)$
provided L is undefined in M

Conflict
$(M; N; U; k; \top) \Rightarrow_{\text{CDCL}} (M; N; U; k; D)$
provided $D \in (N \cup U)$ and $M \models \neg D$

CDCL Rules II

Backtracking Rules

Skip
$(ML^{C \vee L}; N; U; k; D) \Rightarrow_{\text{CDCL}} (M; N; U; k; D)$
provided $D \notin \{\top, \bot\}$ and $\text{comp}(L)$ does not occur in D

Resolve
$(ML^{C \vee L}; N; U; k; D \vee \text{comp}(L)) \Rightarrow_{\text{CDCL}} (M; N; U; k; D \vee C)$
provided D is of level k

Backtrack
$(M_1 K^{i+1} M_2; N; U; k; D \vee L) \Rightarrow_{\text{CDCL}} (M_1 L^{D \vee L}; N; U \cup \{D \vee L\}; i; \top)$
provided L is of level k and D is of level i.

CDCL Properties

Theorem (CDCL Soundness)

CDCL terminates reasonably in two different final states: $(M; N; U; k; \top)$ where $M \models N$ and $(M; N; U; k; \bot)$ where N is unsatisfiable.

Theorem (CDCL Strong Completeness)

For any interpretation M, there is a reasonable sequence of rule applications generating $(M'; N; U; k; \top)$ as a final state, where M and M' only differ in the order of literals.

Theorem (CDCL Termination)

CDCL always terminates reasonably in a state $(M; N; U; k; D)$ with $D \in \{\top, \bot\}$.

Superposition & CDCL

Theorem (N^H_Σ & CDCL)

If $(L_1 \dots L_n; N; U; k; \top)$ is a CDCL state, $\text{atom}(L_i) = P_i$ and $P_1 \prec P_2 \prec \dots \prec P_n$ and $H(P_i) = 1$ if P_i is a decision literal, then $N^H_{P_1, P_2, \dots, P_n}$ contains exactly the atoms from $L_1 \dots L_n$.

CDCL & Redundancy

A learned CDCL clause is the result of a superposition inference and not redundant.

Extension III: Ordering Change

$$\begin{aligned} N \;= \{\; & \neg P_1 \vee P_2 \vee \ldots \vee P_n,\; P_1 \vee P_2 \vee \ldots \vee P_n \\ & \neg P_2 \vee P'_2,\; \neg P_2 \vee \neg P'_2, \\ & \ldots \\ & \neg P_n \vee P'_n,\; \neg P_n \vee \neg P'_n \} \end{aligned}$$

$O(n)$ refutation: $P_n \prec P_{n-1} \ldots \prec P_1 \prec P'_n \prec P'_{n-1} \ldots \prec P'_2$

$O(2^n)$ refutation: $P_n \succ P_{n-1} \ldots \succ P_1 \succ P'_n \succ P'_{n-1} \ldots \succ P'_2$

$$\begin{aligned} N \;= \{\; & Q \vee \neg P_1 \vee P_2 \vee \ldots \vee P_n,\; Q \vee P_1 \vee P_2 \vee \ldots \vee P_n \\ & Q \vee \neg P_2 \vee P'_2,\; Q \vee \neg P_2 \vee \neg P'_2, \\ & \ldots \\ & Q \vee \neg P_n \vee P'_n,\; Q \vee \neg P_n \vee \neg P'_n \} \end{aligned}$$

Consider $N \cup N\{P_i \mapsto P'_i, P'_i \mapsto P_i, Q \mapsto \neg Q\}$

Flexible Model & Redundancy

$N = \{P \vee Q,\; R \vee \neg P\}$ ordering $R \prec Q \prec P$ model $N_{\mathcal{I}} = \{P\}$

$\Rightarrow_{\mathsf{SUP}} Q \vee R$ not redundant

change ordering $P \prec Q \prec R$

now $Q \vee R$ redundant

- flexible ordering not compatible with redundancy
- superposition redundancy is not compatible with CDCL reasoning
- flexible models enable only weaker notions of redundancy
- still: at any point in time, learned CDCL clauses are non-redundant

Be Small

Implementing CDCL

- is subject for an independent tutorial
- like resolution CDCL learns many clauses
- not redundant at creation, but become redundant
- about 10% are subsumed by subsequent learned clause
- checking subsumption for every learned clause is too expensive
- greedily throw away learned clauses by activity heuristic
- slowly increase number of overall kept learned clauses

Summary

Propositional Superposition

- combines static models, fresh learning, redundancy elimination
- explicit ordering
- ordering changes conflict with redundancy

CDCL

- combines flexible models, fresh learning
- implicit ordering
- model changes conflict with redundancy

Future Research

Relationship between flexible models and redundancy.

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